Color dipoles from Bremsstrahlung
in QCD evolution at high energy

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Abstract

We show that the recently developed Hamiltonian theory for high energy evolution in QCD in the dilute regime and in the presence of Bremsstrahlung is consistent with the color dipole picture in the limit where the number of colors N_c is large. The color dipoles are quark–antiquark pairs which can radiate arbitrarily many soft gluons, and the evolution consists in the splitting of any such a dipole into two. We construct the color glass weight function of an onium as a superposition of color dipoles, each represented by a pair of Wilson lines. We show that the action of the Bremsstrahlung Hamiltonian on this weight function and in the large–N_c limit generates the evolution expected from the dipole picture. We construct the dipole number operator in the Hamiltonian theory and deduce the evolution equations for the dipole densities, which are again consistent with the dipole picture. We argue that the Bremsstrahlung effects beyond two gluon emission per dipole are irrelevant for the calculation of scattering amplitudes at high energy.

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1 Introduction

Our purpose in this paper is to demonstrate that the perturbative evolution of a *dilute* hadronic system with increasing energy can be effectively described as the evolution of a system of *color dipoles* provided we consider the limit in which the number of colors N_c is large and that the energy remains low enough for the evolved system to be still dilute. By 'dilute' we mean that the hadronic system is *non–saturated* : the gluon density is low enough for the recombination processes to be unimportant.

At an abstract, *wavefunction*, level, where the evolution is viewed in terms of the partonic content of the system and not of its interactions, the effectiveness of the dipole picture has been of course demonstrated in the original paper by Mueller [1]. Here, however, we are primarily interested in the problem of *scattering* — the dilute hadronic system is the *target* which scatters with an external *projectile*, itself assumed to be dilute — and to that purpose we need to specify not only the evolution of the partons in the target, but also the way how an individual parton (gluon or dipole) couples to the projectile. In previous applications of the dipole picture to scattering [2–7], one has always assumed that a target dipole which partakes in the collision exchanges exactly two gluons — i.e., it undergoes *single scattering* — with the projectile. Whereas this approximation is indeed justified at relatively high energy, where the large dipole density in the target favors the multiple scattering with *different* dipoles [2] (see also the discussion in Sect. 5 below), this is not really correct at low energies, where e.g. the double scattering off a same target dipole via four gluon exchange competes with the scattering off two different dipoles, via twice two gluon exchange.

Notice that multiple gluon exchanges between a target dipole and the projectile correspond to many gluon radiation from the quark and the antiquark legs of the dipole, that is, to *gluon Bremsstrahlung*. The theoretical description of the Bremsstrahlung in the high–energy evolution of a dilute QCD system has recently became available [8–10], but its relation with the dipole picture at large N_c has not been established so far, in spite of previous attempts [11] which emphasized the complexity of the problem. It is our present objective to fully clarify this problem by showing that, at large N_c, the evolution generated by the Bremsstrahlung Hamiltonian in Refs. [8–10] can be recast in the language of the dipole picture.

Specifically, we shall find that, for large N_c, the wavefunction of the dilute target (the 'onium') can be described as a collection of color dipoles, where each dipole is allowed to radiate arbitrarily many small–x gluons in the eikonal approximation. At a mathematical level, a dipole is represented by a pair of Wilson lines (one for the quark and the other one for the antiquark) built with the color field of the radiated gluons. The evolution of the system with increasing energy is governed by the action of the Bremsstrahlung Hamiltonian on the 'onium' wavefunction. As we shall see, at large N_c this evolution proceeds through dipole splitting: one pair of Wilson lines splits into two such pairs which have one common transverse coordinate (at the position of the emitted gluon).

The formulation of the dipole picture that we shall naturally arrive at is of the *color glass* type [12–14] : The dilute target is effectively represented as a stochastic ensemble of classical color fields — the fields radiated by the dipoles — which are distributed according to a functional 'weight function' whose evolution we shall compute. This evolution can be reformulated in terms of probabilities for the dipole configurations (or, equivalently, in terms of dipole n–body densities, with $n \geq 1$), and then it reduces to the original wavefunction evolution[2] by Mueller [1], as expected. But the color glass formulation turns out to be more convenient for applications to the scattering problem, as it gives directly the distribution of the target color fields to which couple the projectile.

At this point we should remind that a color glass formulation of the dipole picture has been already given in Refs. [3, 5], but under the assumption that each dipole can radiate only two gluons. When performing the corresponding approximation on our subsequent results (this amounts to expanding the Wilson lines to lowest non–trivial order), we shall recover the weight function of Refs. [3, 5], as expected. However, this does not mean that the Bremsstrahlung Hamiltonian H_{BREM} should mechanically reduce at large N_c to the 'Dipole Model' Hamiltonian H_{MSW} introduced by Mueller, Shoshi and Wong [5]. Rather, these two Hamiltonians describe the same physical process — the splitting of one dipole into two — but by acting on *different* dipole operators: H_{MSW} acts in the Hilbert space of *bare* dipoles, by which we mean the dipoles which are allowed to radiate only two gluons, whereas H_{BREM} acts on dipoles which are fully *dressed* by the radiation. In fact, the "large–N_c limit of H_{BREM}" is not a well–defined concept by itself: The simplifications appropriate at large N_c can be performed only in the process of acting with H_{BREM} on the onium weight function, or on gauge–invariant correlations expressing dipole densities.

Since both H_{BREM} and H_{MSW} describe dipole splitting (at large N_c), they are equivalent in so far as the evolution of the onium *wavefunction* is concerned: they generate the same evolution equations for the dipole densities (or the associated probabilities), which moreover coincide with the corresponding equations previously derived within the 'abstract' dipole picture [1–4, 16]. However, since the physical information encoded in the dipole operator is different in the two theories — this is richer for H_{BREM}, where a single dipole is allowed to radiate arbitrarily many gluons —, and moreover this information is relevant for the scattering, it follows that the evolution equations for *scattering amplitudes* generated by the two Hamiltonians will be different in general. For instance, the scattering between the onium and two external dipoles may proceed either via twice two gluon exchange with two different internal dipoles, or via four gluon exchange with a single internal dipole. H_{MSW} describes only the former process, and thus generates the 'fluctuation' terms in the evolution equations with Pomeron loops [4, 6], while H_{BREM} describes both of them, and thus in principle it generates more complete equations. Still, as mentioned before, and will be demonstrated in Sect. 5 below, the latter process is suppressed at high energy with respect to the former one, and thus is irrelevant for the study of the approach towards saturation and the unitarity limit. A similar conclusion has been recently reached

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[2] More precisely, the formulation of the 'onium' wavefunction evolution which will naturally emerge from our calculations is that in Refs. [3, 4, 15, 16].

by Marquet, Mueller, Shoshi and Wong [17, 18].

The relation between H_{BREM} and H_{MSW} alluded to above is in fact similar to that between the JIMWLK Hamiltonian and its 'Pomeron' approximation at large N_c, as discussed in Ref. [7]. The JIMWLK Hamiltonian [13, 19, 20] describes non–linear effects (gluon recombination, or merging) in the evolution of the target wavefunction towards saturation. Alternatively, when acting on scattering operators built with Wilson lines, it describes gluon splitting in the projectile followed by the scattering between the products of this splitting and the target. At large N_c, it is convenient to consider a projectile which is itself made with dipoles. Then, as shown in Ref. [7], the action of H_{JIMWLK} on a *dressed* external dipole (where the 'dressing' now refers to multiple scattering, as encoded in the Wilson lines) is equivalent — in the sense of generating the same evolution equations for the scattering amplitudes at large N_c — to that of a simpler 'Pomeron merging' Hamiltonian on a *bare* external dipole, i.e., a dipole which exchanges only two gluons with the target. This close correspondence between the two problems — the JIMWLK evolution in the high density regime and the evolution including Bremsstrahlung in the dilute regime — should not came as a surprise, in view of the *duality* between H_{BREM} and H_{JIMWLK} on one side [9, 10], and between H_{MSW} and the Pomeron–merging Hamiltonian on the other side [7]. Our subsequent analysis will shed more light on this correspondence, by showing that the duality holds, more precisely, between the action of H_{JIMWLK} on *scattering* operators for *external* dipoles and, respectively, that of H_{BREM} on *radiation* operators for the *internal* dipoles. In brief, the action of H_{JIMWLK} on the projectile is dual to that of H_{BREM} on the target.

The plan of the paper is as follows: In Sect. 2 we briefly review the Hamiltonian formulation of the high–energy evolution in the dilute regime and in the presence of Bremsstrahlung, then explain the action of H_{BREM} in terms of Poisson brackets (or 'commutators'), and finally demonstrate as a simple exercise that the action of this Hamiltonian on the 2–point function of the color charge density generates the BFKL equation, as expected. Sect. 3 contains our main results, namely the construction of the onium weight function in terms of dressed dipoles, and the proof of the fact that the action of H_{BREM} on this weight function generates the dipole evolution at large N_c. In Sect. 4, we show how to express dipole number densities in terms of gauge–invariant correlations of the color charge. We introduce the dipole number operator, and verify that, under the action of H_{BREM} (at large N_c), the dipole densities obey the evolution equations expected in the dipole picture. In Sect. 5 we consider the onium scattering with two external dipoles and show that the elementary processes which involve the double scattering of a same target dipole are subdominant at high energy. Finally, Sect. 6 contains our conclusions.

2   QCD evolution in the low density regime: Bremsstrahlung Hamiltonian

The physical problem that we have in mind is that of the scattering between a dilute hadronic system propagating in the positive z (or positive x^+) direction — the right–

moving target — and a system of color dipoles moving in the negative z (or positive x^-) direction — the left–moving projectile. When the projectile is made of a single dipole, the scattering measures the gluon distribution in the target, i.e., the average gluon number density. A projectile made with several dipoles will also probe *fluctuations* in the gluon number. In the limit where the number of colors N_c is large, we shall be able to effectively describe the gluon distribution in the target in terms of (internal) dipoles. Then the scattering with the external system of dipoles will measure the *dipole distribution* in the target, that is, the average dipole number density and the corresponding fluctuations.

The operator expressing the scattering amplitude for an external dipole $(\boldsymbol{x}, \boldsymbol{y})$ in the eikonal approximation reads:

$$T(\boldsymbol{x}, \boldsymbol{y}) = 1 - \frac{1}{N_c} \operatorname{tr}\left(V^\dagger(\boldsymbol{x})V(\boldsymbol{y})\right) \simeq \frac{g^2}{4N_c}\left(\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{y})\right)^2, \tag{2.1}$$

where the second, approximate, equality holds in the case where the target is dilute, which is the case of interest for us here. In this equation, the Wilson line

$$V^\dagger(\boldsymbol{x}) = \operatorname{P}\exp\left\{ig\int dx^- \alpha_a(x^-, \boldsymbol{x})\, t^a\right\}, \tag{2.2}$$

(the t^a's are the generators of the $\mathrm{SU}(N_c)$ algebra in the fundamental representation and the symbol P denotes path–ordering in x^-) describes the eikonal scattering between a quark with transverse coordinate $\boldsymbol{x}$ and the light–cone component $A_a^+ \equiv \alpha_a(x^-, \boldsymbol{x})$ of the color field in the target. Similarly, $V(\boldsymbol{y})$ describes the scattering of the antiquark in the dipole. Furthermore,

$$\alpha_a(\boldsymbol{x}) \equiv \int dx^- \, \alpha_a(x^-, \boldsymbol{x}) \tag{2.3}$$

is the effective color field in the transverse plane, as obtained after integrating over the longitudinal profile of the target, and is related to the corresponding color charge density in the target $\rho_a(\boldsymbol{x})$ via the two–dimensional Poisson equation

$$-\nabla_\perp^2 \alpha_a(\boldsymbol{x}) = \rho_a(\boldsymbol{x}). \tag{2.4}$$

Note that the time variable x^+ is suppressed in the equations above, since the scattering is quasi–instantaneous; thus, α and ρ are the color field and charge at the time of scattering, which is $x^+ = \infty$ in the conventions of Ref. [10] : $\alpha \equiv \alpha_\infty$ and $\rho \equiv \rho_\infty$.

The physical scattering amplitude for a projectile made with k dipoles is obtained as $\langle T^{(k)}\rangle_\tau = \langle T(\boldsymbol{x}_1, \boldsymbol{y}_1)T(\boldsymbol{x}_2, \boldsymbol{y}_2)\cdots T(\boldsymbol{x}_k, \boldsymbol{y}_k)\rangle_\tau$, where $\tau \sim \ln s$ is the rapidity gap between the projectile and the target (note that we use a Lorentz frame in which most of the total energy is carried by the target) and the brackets denote the average over the target wavefunction, which in the spirit of the color glass formalism [14] is computed as an average over the color charge ρ with weight function $Z_\tau[\rho]$. E.g., for a single dipole:

$$\langle T(\boldsymbol{x}, \boldsymbol{y})\rangle_\tau \simeq \int D[\rho] \frac{g^2}{4N_c}\left(\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{y})\right)^2 Z_\tau[\rho]. \tag{2.5}$$

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To account for the correlations induced by Bremsstrahlung, the color charge density should be treated as a time–dependent variable [8–10], so $Z_\tau[\rho]$ is a functional of $\rho_a(x^+, \boldsymbol{x})$ (see below for more details).

According to the above equations, in order to compute physical scattering amplitudes and their evolution with τ, it is sufficient to know the target weight function $Z_\tau[\rho]$ and the corresponding evolution equation. The latter can be compactly written as

$$\frac{\partial}{\partial \tau} Z_\tau[\rho] = - H_{\mathrm{BREM}} Z_\tau[\rho] , \tag{2.6}$$

with the following Hamiltonian for color glass evolution in the dilute regime and in the presence of Bremsstrahlung [8–10] :

$$H_{\mathrm{BREM}} = \frac{1}{(2\pi)^3} \int\limits_{\boldsymbol{xyz}} \mathcal{K}_{\boldsymbol{xyz}} \, \rho_\infty^a(\boldsymbol{x}) \left[1 + \widetilde{W}_{\boldsymbol{x}}\widetilde{W}_{\boldsymbol{y}}^\dagger - \widetilde{W}_{\boldsymbol{x}}\widetilde{W}_{\boldsymbol{z}}^\dagger - \widetilde{W}_{\boldsymbol{z}}\widetilde{W}_{\boldsymbol{y}}^\dagger \right]^{ab} \rho_\infty^b(\boldsymbol{y}). \tag{2.7}$$

(For more clarity, we have temporarily restored the subscript ∞ denoting the x^+ variable of the field ρ.) In this equation,

$$\mathcal{K}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \frac{(\boldsymbol{x} - \boldsymbol{z}) \cdot (\boldsymbol{y} - \boldsymbol{z})}{(\boldsymbol{x} - \boldsymbol{z})^2 (\boldsymbol{z} - \boldsymbol{y})^2} . \tag{2.8}$$

Furthermore, $\widetilde{W}$ and $\widetilde{W}^\dagger$ are all–order differential operators defined as, e.g.,

$$\widetilde{W}(\boldsymbol{x}) = \mathrm{T} \exp \left\{ -g \int dx^+ T^a \frac{\delta}{\delta\rho_a(x^+, \boldsymbol{x})} \right\} \tag{2.9}$$

where T denotes time–ordering and the color matrices T^a are in the adjoint representation. Physically, $\widetilde{W}$ and $\widetilde{W}^\dagger$ are Wilson lines describing the radiation of an arbitrary number of small–x gluons in the eikonal approximation from a single gluon with a larger value of x. (As usual, x denotes the longitudinal momentum fraction carried by a gluon, with $\tau = \ln 1/x$; see Ref. [10] for details.) When Eq. (2.9) is expanded in powers of $\delta/\delta\rho$, each such a power describes the creation of a source for the emission of one gluon (cf. Eq. (2.4)). Since these sources are dilute, the radiated fields are never strong. However, by keeping terms of all orders in $\delta/\delta\rho$ in the Wilson lines, and thus in the Hamiltonian (2.7), one includes in the evolution processes like $2 \to n$ gluon splitting through which the n–point functions of ρ with $n > 2$ get built from the 2–point function $\langle \rho\rho \rangle$ in the dilute regime.

For what follows, the explicit representation (2.9) for $\widetilde{W}$ is not so important. Rather, what matters is the structure of the *Poisson brackets* which define the action of the Hamiltonian (2.7) on the Hilbert space of the operators [3] $\mathcal{O}[\rho_\infty, W]$:

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[3] In what follows, we shall use the notation W for the temporal Wilson line in a generic, or in the fundamental, representations, and we shall keep the more specific notation $\widetilde{W}$ for the adjoint representation.

$$\left[\rho_\infty^a(\boldsymbol{x}), \widetilde{W}_{bc}(\boldsymbol{y})\right] = g(T^a\widetilde{W}(\boldsymbol{x}))_{bc}\,\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}),$$
$$\left[\rho_\infty^a(\boldsymbol{x}), \rho_\infty^b(\boldsymbol{y})\right] = -igf^{abc}\rho_\infty^c(\boldsymbol{x})\,\delta^{(2)}(\boldsymbol{x}-\boldsymbol{y}),$$
$$\left[\widetilde{W}_{ab}(\boldsymbol{x}), \widetilde{W}_{cd}(\boldsymbol{y})\right] = 0\,. \tag{2.10}$$

The first two equations above show that the color charges ρ_∞^a act as infinitesimal gauge rotations of the Wilson line W at its end points, that is, as Lie derivatives. One can check that these commutation relations satisfy the right properties expected for Poisson brackets, in particular, they obey the Jacobi identity.

The non–commutativity of the color charge variables $\rho_\infty^a(\boldsymbol{x})$ with themselves is a source of potential difficulties (like ambiguities in the ordering of the operators) in the construction of the correlation functions and of the color glass weight function [8–11]. Still, as we shall see, this problem can be systematically avoided in the large–N_c limit, where we shall be able to explicitly construct the weight function $Z_\tau[\rho]$ in a way which is free of ambiguities.

The formal evolution equation for some operator $\mathcal{O}[\rho_\infty, W]$ is defined by its Poisson bracket with H_{BREM} :

$$\frac{\partial}{\partial\tau}\,\mathcal{O}[\rho_\infty, W] = \left[H_{\mathrm{BREM}}, \mathcal{O}\right]. \tag{2.11}$$

Note that for a given ordering of the operators which compose $\mathcal{O}$, the operation above is unambiguous. However, its result will generally change when we permute the operators within the definition of $\mathcal{O}$. Once again, this difficulty will not show up at large N_c.

One can check that, for physical observables at least, Eq. (2.11) is indeed consistent with the equation (2.6) for the evolution of the weight function together with the definition (2.5) of the color glass average. To see this, note first that the physical observables are gauge–invariant, which in the present context means that they are invariant under the gauge transformations dependent upon x^+ [10]. When the action of H_{BREM} is restricted to such gauge–invariant operators, Eq. (2.7) can be equivalently replaced by [21]

$$H_{\mathrm{BREM}} = \frac{-1}{16\pi^3}\int\limits_{\boldsymbol{xyz}}\mathcal{M}_{\boldsymbol{xyz}}\left[1 + \widetilde{W}_{\boldsymbol{x}}\widetilde{W}_{\boldsymbol{y}}^\dagger - \widetilde{W}_{\boldsymbol{x}}\widetilde{W}_{\boldsymbol{z}}^\dagger - \widetilde{W}_{\boldsymbol{z}}\widetilde{W}_{\boldsymbol{y}}^\dagger\right]^{ab}\rho_\infty^a(\boldsymbol{x})\rho_\infty^b(\boldsymbol{y}), \tag{2.12}$$

with $\mathcal{M}_{\boldsymbol{xyz}}$ denoting the dipole kernel [1] :

$$\mathcal{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} = \mathcal{K}_{\boldsymbol{xxz}} + \mathcal{K}_{\boldsymbol{yyz}} - 2\mathcal{K}_{\boldsymbol{xyz}}\,. \tag{2.13}$$

(This can be proven through the 'dual' version of the arguments used in Ref. [21] for the case of the JIMWLK Hamiltonian.) From Eq. (2.10) one can check that, in the presence of the dipole kernel, the color charge operators in Eq. (2.12) can be freely commuted through the Wilson lines there; in writing Eq. (2.12) we have used this freedom to commute both factors of ρ fully to the right, which is convenient for the subsequent manipulations. Also,

the non–commutativity between $\rho_\infty^a(\boldsymbol{x})$ and $\rho_\infty^b(\boldsymbol{y})$ plays no role in Eq. (2.12) since the Wilson–line part of the integrand,

$$h_{ab}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \left[1 + \widetilde{W}_{\boldsymbol{x}} \widetilde{W}_{\boldsymbol{y}}^\dagger - \widetilde{W}_{\boldsymbol{x}} \widetilde{W}_{\boldsymbol{z}}^\dagger - \widetilde{W}_{\boldsymbol{z}} \widetilde{W}_{\boldsymbol{y}}^\dagger \right]^{ab}, \tag{2.14}$$

is symmetric under the simultaneous exchange $a \leftrightarrow b$ and $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$.

Note furthermore that, if the operator $\mathcal{O}[\rho_\infty, W]$ contains a factor of W on the left of all the other operators, then when computing the color glass average $\langle \mathcal{O}[\rho_\infty, W] \rangle_\tau$ this factor can be replaced by one. Indeed, when expanding the exponential in Eq. (2.9), all the terms but the first one yield total derivatives which vanish after integration over ρ. In the subsequent manipulations, it will be often convenient to 'normal–order' the operators by pushing factors of W all the way to the left and the replacing them by one when computing the average.

We are now prepared to check that Eqs. (2.6) and (2.11) are consistent with each other. To that aim, take the color glass average in Eq. (2.11). This involves

$$\int D[\rho] \left[h_{ab}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \rho_\infty^a(\boldsymbol{x}) \rho_\infty^b(\boldsymbol{y}), \mathcal{O} \right] Z_\tau[\rho]$$

$$= \int D[\rho] \left(h_{ab} \rho_\infty^a \rho_\infty^b \mathcal{O} - \mathcal{O} h_{ab} \rho_\infty^a \rho_\infty^b \right) Z_\tau[\rho]$$

$$= \int D[\rho] \left(- \mathcal{O} h_{ab} \rho_\infty^a \rho_\infty^b \right) Z_\tau[\rho] \longrightarrow \int D[\rho] \, \mathcal{O} \left(- H_{\text{BREM}} Z_\tau[\rho] \right), \tag{2.15}$$

where we have used the fact that $h[W] \to 0$ when $W \to 1$. By the same argument, we deduce that the average of Eq. (2.11) involves only the commutator $[h, \mathcal{O}]$:

$$\frac{\partial}{\partial \tau} \langle \mathcal{O} \rangle_\tau = \frac{-1}{16\pi^3} \int D[\rho] \int_{\boldsymbol{xyz}} \mathcal{M}_{\boldsymbol{xyz}} \left[h_{ab}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \mathcal{O} \right] \rho_\infty^a(\boldsymbol{x}) \rho_\infty^b(\boldsymbol{y}) \, Z_\tau[\rho]. \tag{2.16}$$

As a first, relatively simple, application of the above formalism, let us derive in this way the BFKL equation. We shall argue later that, at large N_c and for $\boldsymbol{x} \neq \boldsymbol{y}$, the charge–charge correlator $\langle \rho^a(\boldsymbol{x}) \rho^a(\boldsymbol{y}) \rangle_\tau$ is proportional to the dipole number density $n_\tau(\boldsymbol{x}, \boldsymbol{y})$, to be precisely defined in Sect. 4. Here and in what follows, we use the simpler notation $\rho_{\boldsymbol{x}}^a \equiv \rho_\infty^a(\boldsymbol{x})$. Specifically :

$$\langle \rho^a(\boldsymbol{x}) \rho^a(\boldsymbol{y}) \rangle_\tau = -g^2 C_F \left[n_\tau(\boldsymbol{x}, \boldsymbol{y}) + n_\tau(\boldsymbol{y}, \boldsymbol{x}) \right] \quad \text{for} \quad \boldsymbol{x} \neq \boldsymbol{y}. \tag{2.17}$$

Since, on the other hand, the BFKL equation for a 2–point function is well known to emerge independently of the large–N_c approximation, we expect the quantity in Eq. (2.17) to obey the BFKL equation for arbitrary N_c. Let us check that this is indeed the case. According to Eq. (2.16), we have:

$$\frac{\partial}{\partial \tau} \langle \rho^a(\boldsymbol{x}) \rho^a(\boldsymbol{y}) \rangle_\tau = \frac{-1}{16\pi^3} \int_{\boldsymbol{uvz}} \mathcal{M}_{\boldsymbol{uvz}} \left\langle \left[h_{cd}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}), \rho_{\boldsymbol{x}}^a \rho_{\boldsymbol{y}}^a \right] \rho_{\boldsymbol{u}}^c \rho_{\boldsymbol{v}}^d \right\rangle_\tau, \tag{2.18}$$

where we have relabeled the transverse coordinates internal to H_{BREM} as $\boldsymbol{u},$, $\boldsymbol{v}$ and $\boldsymbol{z}$. By repeated use of the commutation relation, we shall move h_{cd} (which contains the Wilson lines W) to the left and then set $W = 1$

$$
\begin{aligned}
[h_{cd}, \rho_{\boldsymbol{x}}^a \rho_{\boldsymbol{y}}^a] &= [h_{cd}, \rho_{\boldsymbol{x}}^a]\rho_{\boldsymbol{y}}^a + \rho_{\boldsymbol{x}}^a [h_{cd}, \rho_{\boldsymbol{y}}^a] \\
&= [h_{cd}, \rho_{\boldsymbol{x}}^a]\rho_{\boldsymbol{y}}^a + [h_{cd}, \rho_{\boldsymbol{y}}^a]\rho_{\boldsymbol{x}}^a + [\, \rho_{\boldsymbol{x}}^a, \, [h_{cd}, \rho_{\boldsymbol{y}}^a]\,] \\
&\longrightarrow [\, \rho_{\boldsymbol{x}}^a, \, [h_{cd}, \rho_{\boldsymbol{y}}^a]\,],
\end{aligned}
\tag{2.19}
$$

where, as shown in the last line, only the double commutator must be retained when computing the color glass average in Eq. (2.18). Indeed

$$
[h_{cd}, \rho_{\boldsymbol{x}}^a]\big|_{W=1} = 0 \,,
\tag{2.20}
$$

as it can be easily checked by using Eq. (2.14) together with the following commutator, which in turn follows from Eq. (2.10) ($\delta_{\boldsymbol{ux}} \equiv \delta^{(2)}(\boldsymbol{u} - \boldsymbol{x})$) :

$$
[\, \rho_{\boldsymbol{x}}^a, (\widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger)_{cd}\,] = g\delta_{\boldsymbol{ux}}\big(T^a \widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger\big)_{cd} - g\delta_{\boldsymbol{vx}}\big(\widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger T^a\big)_{cd}.
\tag{2.21}
$$

To evaluate the right hand side of Eq. (2.18) we also need

$$
\begin{aligned}
\Big[\rho_{\boldsymbol{x}}^a, \big[\rho_{\boldsymbol{y}}^b, (\widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger)_{cd}\big]\Big] &= g^2 \delta_{\boldsymbol{uy}} \Big\{ \delta_{\boldsymbol{ux}}\big(T^b T^a \widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger\big)_{cd} - \delta_{\boldsymbol{vx}}\big(T^b \widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger T^a\big)_{cd} \Big\} \\
&\quad - g^2 \delta_{\boldsymbol{vy}} \Big\{ \delta_{\boldsymbol{ux}}\big(T^a \widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger T^b\big)_{cd} - \delta_{\boldsymbol{vx}}\big(\widetilde{W}_{\boldsymbol{u}} \widetilde{W}_{\boldsymbol{v}}^\dagger T^a T^b\big)_{cd} \Big\} \,.
\end{aligned}
\tag{2.22}
$$

From now on, simple algebra yields

$$
\frac{\partial}{\partial \tau} \langle \rho^a(\boldsymbol{x})\rho^a(\boldsymbol{y})\rangle_\tau = \frac{g^2 N_c}{8\pi^3} \int_{\boldsymbol{uvz}} \mathcal{M}_{\boldsymbol{uvz}} \big(-\delta_{\boldsymbol{ux}}\delta_{\boldsymbol{vy}} + \delta_{\boldsymbol{ux}}\delta_{\boldsymbol{zy}} + \delta_{\boldsymbol{zx}}\delta_{\boldsymbol{vy}} \big) \langle \rho_{\boldsymbol{u}}^c \rho_{\boldsymbol{v}}^c\rangle_\tau, \tag{2.23}
$$

where we have neglected the terms which vanish at $\boldsymbol{x} \neq \boldsymbol{y}$ and used the symmetry $\boldsymbol{u} \leftrightarrow \boldsymbol{v}$ of the kernel in the Hamiltonian. After also using Eq. (2.17), this is finally rewritten as

$$
\begin{aligned}
\frac{\partial n_\tau(\boldsymbol{x}, \boldsymbol{y})}{\partial \tau} = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \Big[&-\mathcal{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\, n_\tau(\boldsymbol{x}, \boldsymbol{y}) \\
&+ \mathcal{M}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y})\, n_\tau(\boldsymbol{x}, \boldsymbol{z}) + \mathcal{M}(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x})\, n_\tau(\boldsymbol{z}, \boldsymbol{y}) \Big]
\end{aligned}
\tag{2.24}
$$

(with $\bar{\alpha}_s = \alpha_s N_c/\pi$), which at large N_c is recognized as the BFKL equation for the dipole number density [4, 16], but which is valid as written for arbitrary N_c.

Note that the same equation could have been obtained by first expanding the Wilson lines in the Hamiltonian (2.7) in a power series in derivatives, then keeping the first non–trivial terms (the second order ones) in this expansion to deduce the BFKL Hamiltonian:

$$
H_{\text{BFKL}} = \frac{-g^2}{16\pi^3} \int_{\boldsymbol{uvz}} \mathcal{M}_{\boldsymbol{uvz}} f^{ace} f^{bde} \left[\frac{\delta}{\delta\rho^c(\boldsymbol{u})} - \frac{\delta}{\delta\rho^c(\boldsymbol{z})} \right] \left[\frac{\delta}{\delta\rho^d(\boldsymbol{z})} - \frac{\delta}{\delta\rho^d(\boldsymbol{v})} \right] \rho^a(\boldsymbol{u})\rho^b(\boldsymbol{v}),
$$
$$
\tag{2.25}
$$

9

and finally using this Hamiltonian in the evolution equation for $\langle \rho^a(\boldsymbol{x})\rho^a(\boldsymbol{y})\rangle_\tau$:

$$\frac{\partial}{\partial\tau}\langle \rho^a(\boldsymbol{x})\rho^a(\boldsymbol{y})\rangle_\tau \,=\, \int D[\rho]\,\rho^a(\boldsymbol{x})\rho^a(\boldsymbol{y})\big(-H_{\mathrm{BFKL}}\,Z_\tau[\rho]\big)\,. \tag{2.26}$$

After some integration by parts, the functional derivatives in Eq. (2.25) are brought to act on the factors of ρ in the operator, and then Eq. (2.23) immediately follows.

One may think that the first derivation of Eq. (2.23), in which the Wilson lines inside H_{BREM} were kept unexpanded, is more general than the one based on the BFKL Hamiltonian (2.25), but this is only illusory: Since $\mathcal{O} = \rho_{\boldsymbol{x}}^a\rho_{\boldsymbol{y}}^a$ is quadratic in ρ, all the higher order ρ–derivatives beyond the second–order ones kept in Eq. (2.25) do not contribute to its evolution. Thus, in so far as Eq. (2.23) is concerned, the two methods presented above — the use of the commutation relations and the derivative expansion of the BREM Hamiltonian — are equivalent with each other. However, for more general situations (e.g., more complicated observables, or the evolution of the weight function that we shall consider in the next section), the use of the commutation relations turns out to be more convenient as it avoids potential ambiguities with the x^+–ordering of the functional derivatives in the expansion of the Wilson line (2.9).

3   Dipole picture from the Bremsstrahlung Hamiltonian

In this section we shall derive the dipole picture from the action of the Bremsstrahlung Hamiltonian in the large–N_c limit. More precisely, we shall show that, if one starts with a single color dipole — a quark–antiquark pair with the quark at $\boldsymbol{u}_0$ and the antiquark at $\boldsymbol{v}_0$ — at the initial rapidity $\tau_0 = 0$, then the partonic system produced at some higher rapidity τ through the evolution described by H_{BREM} at large N_c can be itself characterized as a collection of $q\bar{q}$ color dipoles (an 'onium'), which evolves through dipole splitting. Besides the Bremsstrahlung Hamiltonian (2.7), the crucial ingredient in this picture is the description of the onium wavefunction as a *color glass*, which in turn requires the proper definition of the dipole operator as a *color source*.

The first formulation of the onium as a color glass has been given in Ref. [3], under the assumption that each dipole is a color source for only two gluons. The evolution of the associated weight function through gluon splitting has been exhibited in Ref. [3], but it was only later, in Ref. [4], that one has realized that this evolution cannot be fully accounted for by the JIMWLK Hamiltonian. The appropriate Hamiltonian has been constructed shortly after, by Mueller, Shoshi and Wong [5]. It is the sum of the BFKL Hamiltonian (2.25) plus a term involving four ρ–derivatives which describes dipole splitting (again, under the assumption that each dipole can radiate only two gluons). In what follows, we shall generalize the construction in Refs. [3, 5] to the case where a dipole can radiate arbitrarily many (small–x) gluons. That is, we shall construct the corresponding dipole operator and onium weight function, and show that the Bremsstrahlung Hamiltonian (2.7) is the appropriate generalization of the MSW Hamiltonian [5].

We start with a brief summary of the results in Refs. [3, 5], to which we shall refer as the *'Dipole Model'* (DM). The onium (color–glass) weight function in the DM reads:

$$Z_\tau^{\mathrm{DM}}[\rho] \;=\; \sum_{N=1}^{\infty} \int d\Gamma_N \, P_N(\{z_i\}; \tau) \prod_{i=1}^{N} D_0^\dagger(z_{i-1}, z_i) \, \delta[\rho] \,, \tag{3.1}$$

where $P_N(\{z_i\}; \tau)$ denotes the probability density to find a given configuration of N dipoles at rapidity τ (the configuration being specified by $N - 1$ transverse coordinates $\{z_i\} = \{z_1, z_2, \dots z_{N-1}\}$, such that the coordinates of the N dipoles are (z_0, z_1), (z_1, z_2),...,(z_{N-1}, z_N), with $z_0 \equiv u_0$ and $z_N \equiv v_0$), and $d\Gamma_N$ denotes the measure for the phase–space integration: $d\Gamma_N = \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \dots \mathrm{d}^2 z_{N-1}$. Furthermore, $D_0^\dagger(u, v)$ is the DM dipole creation operator, with the dipole assimilated to the source of two gluons:

$$D_0^\dagger(x, y) \;\equiv\; 1 + \frac{g^2}{4N_c} \left(\frac{\delta}{\delta \rho^a(x)} - \frac{\delta}{\delta \rho^a(y)} \right)^2 , \tag{3.2}$$

where $\rho^a(x) = \int dx^- \rho^a(x^-, x)$ should be interpreted as the color charge density in the transverse plane at the interaction time. (There is no explicit x^+–dependence in the DM picture; see also Eq. (3.9) below.) Finally, the delta–functional $\delta[\rho] \equiv \delta[\rho^a(x)]$ is defined in the context of the functional integral over $\rho^a(x)$.

The evolution of the DM is driven by

$$\frac{\partial}{\partial \tau} Z_\tau^{\mathrm{DM}}[\rho] \;=\; - H_{\mathrm{MSW}} \, Z_\tau^{\mathrm{DM}}[\rho] \,, \tag{3.3}$$

with the Mueller–Shoshi–Wong Hamiltonian

$$H_{\mathrm{MSW}} \;=\; -\frac{\bar{\alpha}_s}{2\pi} \int_{xyz} \mathcal{M}(x, y, z) \Big[- D_0^\dagger(x, y) + D_0^\dagger(x, z) D_0^\dagger(z, y) \Big] D_0(x, y), \tag{3.4}$$

where $D_0(x, y)$ is the dipole annihilation operator within the DM:

$$D_0(x, y) \;=\; -\frac{1}{g^2 N_c} \, \rho^a(x) \rho^a(y) \quad \text{for} \quad x \neq y. \tag{3.5}$$

One can check indeed that:

$$[\, D_0(x, y), D_0^\dagger(u, v) \,] \;\approx\; \frac{1}{2} \Big(\delta_{ux} \delta_{vy} + \delta_{uy} \delta_{vx} \Big), \tag{3.6}$$

where the approximate equality sign means that the equality holds in the large–N_c limit. (In Sect. 4, we shall demonstrate a relation similar to Eq. (3.6) in a more general context.)

By inserting Eqs. (3.1) and (3.4) into Eq. (3.3), then using Eq. (3.6) to successively commute the annihilation operator $D_0(x, y)$ from H_{MSW} to the right of the creation operators $D_0^\dagger(z_{i-1}, z_i)$ from Z_τ^{DM}, and finally using the fact that $D_0 \delta[\rho] = 0$, one finds the following evolution equation for the onium weight function:

$$\frac{\partial}{\partial\tau} Z_\tau^{\mathrm{DM}}[\rho] \approx \frac{\bar{\alpha}_s}{2\pi} \sum_{N=1}^\infty \int d\Gamma_N \, P_N(\tau) \sum_{i=1}^N \int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{z}_{i-1}, \boldsymbol{z}_i, \boldsymbol{z})$$
$$\times \Big[D_0^\dagger(\boldsymbol{z}_{i-1}, \boldsymbol{z}) D_0^\dagger(\boldsymbol{z}, \boldsymbol{z}_i) - D_0^\dagger(\boldsymbol{z}_{i-1}, \boldsymbol{z}_i) \Big] \prod_{j \neq i} D_0^\dagger(\boldsymbol{z}_{j-1}, \boldsymbol{z}_j) \, \delta[\rho], \qquad (3.7)$$

which is the expression of the dipole picture in the color glass representation [3]: the evolution of the weight function proceeds through dipole splitting. From here on, one can proceed in the standard way [3, 4, 16] to deduce evolution equations for the dipole densities and probabilities (see the discussion at the end of this section and in Sect. 4).

We now return to the general case where a dipole can emit an arbitrary number of gluons, and show that an evolution equation similar to Eq. (3.7) (which is synonymous of the dipole picture) is obtained also in that case provided we replace H_{MSW} by H_{BREM} and the 'bare dipole' creation operator $D_0^\dagger$ by the following, 'dressed dipole', operator [11] :

$$D^\dagger(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{N_c} \mathrm{tr}\Big(W(\boldsymbol{x}) W^\dagger(\boldsymbol{y}) \Big), \qquad (3.8)$$

where the Wilson lines are in the fundamental representation (W stays for the quark, and $W^\dagger$ for the antiquark). To second order in the expansion of the Wilson lines in powers of $\delta/\delta\rho$, and with the following identification

$$\frac{\delta}{\delta\rho^a(\boldsymbol{x})} \equiv \int dx^+ \frac{\delta}{\delta\rho^a(x^+, \boldsymbol{x})}, \qquad (3.9)$$

Eq. (3.8) reduces to the two–gluon emission operator, Eq. (3.2). In general, Eq. (3.8) can be seen as the gauge–invariant generalization of Eq. (3.2) to the regime where the derivatives are formally strong : $g \int dx^+ (\delta/\delta\rho) \sim 1$ (cf. the discussion after Eq. (2.9)).

The onium weight function will be now constructed by analogy with Eq. (3.1). That is, we start by assuming that the weight function $Z_\tau[\rho]$ in the large–N_c limit can be cast in the following, dipolar, form :

$$Z_\tau[\rho] = \sum_{N=1}^\infty \int d\Gamma_N \, P_N(\{\boldsymbol{z}_i\}; \tau) \prod_{i=1}^N D^\dagger(\boldsymbol{z}_{i-1}, \boldsymbol{z}_i) \, \delta[\rho], \qquad (3.10)$$

and then show that this particular structure is indeed preserved by the evolution under the action of the Bremsstrahlung Hamiltonian (2.7) and for large N_c.

The subsequent mathematical manipulations can perhaps be better understood if one notices their analogy (in fact, *duality*) to manipulations which are by now familiar in the context of the JIMWLK evolution (see, e.g., Refs. [13, 21]). Specifically, the Bremsstrahlung and JIMWLK evolutions are known to be *dual* to each other [9, 10], in the sense that the corresponding Hamiltonians and also the respective Poisson brackets get interchanged with each other under the following duality transformations [4] :

---

[4] Note that the "∞" subscript in α_∞ refers to x^-, and not to x^+; see Ref. [10] for details.

$$\frac{1}{i}\frac{\delta}{\delta\alpha_\infty^a(\boldsymbol{x})} \longleftrightarrow \rho_\infty^a(\boldsymbol{x})\,, \qquad V^\dagger(\boldsymbol{x}) \longleftrightarrow W(\boldsymbol{x})\,. \tag{3.11}$$

Similarly, the dipole creation operator (3.8) is dual to the S–matrix operator $S(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{N_c}\mathrm{tr}(V^\dagger(\boldsymbol{x})V(\boldsymbol{y}))$ which describes the scattering of an external dipole in the high–density regime specific to the JIMWLK evolution (cf. Eq. (2.1)). Thus, clearly, H_{BREM} acts on the *internal* dipoles in the same way as H_{JIMWLK} does on the *external* ones. We know already that the action of H_{JIMWLK} on scattering operators generates the Balitsky equations [22]. At large N_c and for projectiles built with dipoles, the Balitsky equations close in the space of dipole operators, and are consistent with the dipole picture for the *projectile* wavefunction [4, 16, 23]. This lets us anticipate that the action of H_{BREM} on the dipolar weight function (3.10) at large N_c should similarly generate the dipole picture for the *target* wavefunction. This will be verified explicitly in what follows.

We thus need to evaluate

$$\frac{\partial}{\partial\tau}Z_\tau[\rho] = -H_{\mathrm{BREM}}Z_\tau[\rho]$$

$$= \sum_{N=1}^{\infty}\int d\Gamma_N\,P_N(\tau)\frac{1}{16\pi^3}\int_{\boldsymbol{xyz}}\mathcal{M}_{\boldsymbol{xyz}}\,h_{ab}\rho_{\boldsymbol{x}}^a\rho_{\boldsymbol{y}}^b\prod_{i=1}^{N}D^\dagger(\boldsymbol{z}_{i-1}, \boldsymbol{z}_i)\,\delta[\rho] \tag{3.12}$$

in the large–N_c limit. For more clarity, it is convenient to use the notations $\boldsymbol{u}_i \equiv \boldsymbol{z}_{i-1}$ and $\boldsymbol{v}_i \equiv \boldsymbol{z}_i$ for the transverse coordinates of the i–th dipole; it is then understood that $\boldsymbol{u}_{i+1} = \boldsymbol{v}_i$. Let us focus on the action of H_{BREM} on the weight function Z_N for a given configuration of N dipoles. This involves:

$$h_{ab}\rho_{\boldsymbol{x}}^a\rho_{\boldsymbol{y}}^b\,Z_N(\{\boldsymbol{u}_i, \boldsymbol{v}_i\}) \equiv h_{ab}\rho_{\boldsymbol{x}}^a\rho_{\boldsymbol{y}}^b\prod_{i=1}^{N}D^\dagger(\boldsymbol{u}_i, \boldsymbol{v}_i)\,\delta[\rho]\,. \tag{3.13}$$

To proceed, we need to commute $\rho_{\boldsymbol{x}}^a\rho_{\boldsymbol{y}}^b$ to the right of the dipole creation operators and then use $\rho\,\delta[\rho] \equiv 0$. The relevant commutator is

$$\left[\rho_{\boldsymbol{x}}^a\rho_{\boldsymbol{y}}^b, D_{\boldsymbol{uv}}^\dagger\right] = \left[\rho_{\boldsymbol{x}}^a, D_{\boldsymbol{uv}}^\dagger\right]\rho_{\boldsymbol{y}}^b + \left[\rho_{\boldsymbol{y}}^b, D_{\boldsymbol{uv}}^\dagger\right]\rho_{\boldsymbol{x}}^a + \left[\rho_{\boldsymbol{x}}^a, \left[\rho_{\boldsymbol{y}}^b, D_{\boldsymbol{uv}}^\dagger\right]\right] \tag{3.14}$$

By adapting Eqs. (2.21)–(2.22) to the fundamental representation, we obtain

$$\left[\rho_{\boldsymbol{x}}^a\rho_{\boldsymbol{y}}^b, D_{\boldsymbol{uv}}^\dagger\right] = \frac{g}{N_c}(\delta_{\boldsymbol{xu}} - \delta_{\boldsymbol{xv}})\mathrm{tr}(t^a W_{\boldsymbol{u}}W_{\boldsymbol{v}}^\dagger)\,\rho_{\boldsymbol{y}}^b$$

$$+\frac{g}{N_c}(\delta_{\boldsymbol{yu}} - \delta_{\boldsymbol{yv}})\mathrm{tr}(t^b W_{\boldsymbol{u}}W_{\boldsymbol{v}}^\dagger)\,\rho_{\boldsymbol{x}}^a$$

$$+g^2(\delta_{\boldsymbol{yu}} - \delta_{\boldsymbol{yv}})\left[\delta_{\boldsymbol{xu}}\frac{1}{N_c}\mathrm{tr}(W_{\boldsymbol{u}}W_{\boldsymbol{v}}^\dagger t^b t^a) - \delta_{\boldsymbol{xv}}\frac{1}{N_c}\mathrm{tr}(W_{\boldsymbol{u}}W_{\boldsymbol{v}}^\dagger t^a t^b)\right], \tag{3.15}$$

where the expression in the last line represents the double commutator term. We shall shortly verify that the first two terms on the right hand side (proportional to ρ), when

multiplied by h_{ab} and acting on Z_N, give subleading contributions at large N_c compared to the last term, which does not involve ρ. Introducing the more compact notation

$$
\begin{aligned}
\left[\rho_{\boldsymbol{x}}^a \rho_{\boldsymbol{y}}^b, D_{\boldsymbol{uv}}^\dagger\right]_{\text{non-dipole}} &\equiv \frac{g}{N_c}(\delta_{\boldsymbol{xu}} - \delta_{\boldsymbol{xv}})\text{tr}(t^a W_{\boldsymbol{u}} W_{\boldsymbol{v}}^\dagger)\, \rho_{\boldsymbol{y}}^b \\
&\quad + \frac{g}{N_c}(\delta_{\boldsymbol{yu}} - \delta_{\boldsymbol{yv}})\text{tr}(t^b W_{\boldsymbol{u}} W_{\boldsymbol{v}}^\dagger)\, \rho_{\boldsymbol{x}}^a \\
&\equiv A_{\boldsymbol{uvx}}^a\, \rho_{\boldsymbol{y}}^b + B_{\boldsymbol{uvy}}^b\, \rho_{\boldsymbol{x}}^a,
\end{aligned}
\tag{3.16}
$$

where A and B do not contain ρ, we can write the result of the first commutation as follows (in simplified notations whose meaning should be obvious) :

$$
\begin{aligned}
\rho_{\boldsymbol{x}}^a \rho_{\boldsymbol{y}}^b D_1^\dagger D_2^\dagger \ldots D_N^\dagger &= D_1^\dagger\, \rho_{\boldsymbol{x}}^a \rho_{\boldsymbol{y}}^b\, D_2^\dagger \ldots D_N^\dagger + \left[\rho_{\boldsymbol{x}}^a \rho_{\boldsymbol{y}}^b, D_{\boldsymbol{u}_1 \boldsymbol{v}_1}^\dagger\right]_{\text{non-dipole}} D_2^\dagger \ldots D_N^\dagger \\
&\quad + \left[\rho_{\boldsymbol{x}}^a, \left[\rho_{\boldsymbol{y}}^b, D_{\boldsymbol{u}_1 \boldsymbol{v}_1}^\dagger\right]\right] D_2^\dagger \ldots D_N^\dagger.
\end{aligned}
\tag{3.17}
$$

The last, double–commutator, term in the r.h.s. of the above equation is already independent of ρ, so we can operate with h_{ab} on it :

$$
\begin{aligned}
\frac{1}{16\pi^3} \int\limits_{\boldsymbol{xyz}} \mathcal{M}_{\boldsymbol{xyz}}\, h_{ab}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &\left[\rho_{\boldsymbol{x}}^a, \left[\rho_{\boldsymbol{y}}^b, D_{\boldsymbol{u}_1 \boldsymbol{v}_1}^\dagger\right]\right] = \\
&= -\frac{g^2}{8\pi^3 N_c} \int\limits_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{u}_1 \boldsymbol{v}_1 \boldsymbol{z}}\, h_{ab}(\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{z})\, \text{tr}(W_{\boldsymbol{u}_1} W_{\boldsymbol{v}_1}^\dagger t^b t^a) = \\
&= \frac{\bar{\alpha}_s}{2\pi} \int\limits_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{u}_1 \boldsymbol{v}_1 \boldsymbol{z}} \left\{ -\frac{1}{N_c}\text{tr}(W_{\boldsymbol{u}_1} W_{\boldsymbol{v}_1}^\dagger) + \frac{1}{N_c}\text{tr}(W_{\boldsymbol{u}_1} W_{\boldsymbol{z}}^\dagger) \frac{1}{N_c}\text{tr}(W_{\boldsymbol{z}} W_{\boldsymbol{v}_1}^\dagger) \right\}
\end{aligned}
\tag{3.18}
$$

where we have also used the identity $(t^a)_{ij}(t^a)_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta_{kl}$. The last equation is dual to the first Balitsky equation [22], and in fact this has been obtained here through manipulations similar to those usually performed in the derivation of the Balitsky hierarchy from the JIMWLK equation [13]. The first, negative, term within the braces in Eq. (3.18) describes the probability that the original dipole $(\boldsymbol{u}_1, \boldsymbol{v}_1)$ survive without splitting, while the second, positive, term describes the splitting of the original dipole into the new dipoles $(\boldsymbol{u}_1, \boldsymbol{z})$ and $(\boldsymbol{z}, \boldsymbol{v}_1)$.

We shall now check that the non–dipolar contribution to Eq. (3.17) is indeed suppressed at large N_c. To that aim, we take one particular piece in the non–dipolar commutator (3.16), say the first piece $A_{\boldsymbol{u}_1 \boldsymbol{v}_1 \boldsymbol{x}}^a \rho_{\boldsymbol{y}}^b$, and consider its action on the dipole creation operators which appear on its right in Eq. (3.17):

$$
\begin{aligned}
A_{\boldsymbol{u}_1 \boldsymbol{v}_1 \boldsymbol{x}}^a\, \rho_{\boldsymbol{y}}^b D_{\boldsymbol{u}_2 \boldsymbol{v}_2}^\dagger D_{\boldsymbol{u}_3 \boldsymbol{v}_3}^\dagger \ldots D_{\boldsymbol{u}_N \boldsymbol{v}_N}^\dagger\, \delta[\rho] &= D_{\boldsymbol{u}_2 \boldsymbol{v}_2}^\dagger A_{\boldsymbol{u}_1 \boldsymbol{v}_1 \boldsymbol{x}}^a\, \rho_{\boldsymbol{y}}^b D_{\boldsymbol{u}_3 \boldsymbol{v}_3}^\dagger \ldots D_{\boldsymbol{u}_N \boldsymbol{v}_N}^\dagger\, \delta[\rho] \\
&\quad + \left[A_{\boldsymbol{u}_1 \boldsymbol{v}_1 \boldsymbol{x}}^a\, \rho_{\boldsymbol{y}}^b, D_{\boldsymbol{u}_2 \boldsymbol{v}_2}^\dagger\right] D_{\boldsymbol{u}_3 \boldsymbol{v}_3}^\dagger \ldots D_{\boldsymbol{u}_N \boldsymbol{v}_N}^\dagger\, \delta[\rho].
\end{aligned}
\tag{3.19}
$$

The above commutator is independent of ρ, and the same is true for all the other commutators generated when ρ^b is further commuted towards the right. It is therefore sufficient

to evaluate the action of h_{ab} on one such a commutator. We have:

$$\left[A^a_{\boldsymbol{u_1 v_1 x}}\,\rho^b_{\boldsymbol{y}}, D^\dagger_{\boldsymbol{u_2 v_2}}\right] = \frac{g}{N_c}(\delta_{\boldsymbol{xu_1}} - \delta_{\boldsymbol{xv_1}})\mathrm{Tr}(t^a W_{\boldsymbol{u_1}} W^\dagger_{\boldsymbol{v_1}})\left[\rho^b_{\boldsymbol{y}}, D^\dagger_{\boldsymbol{u_2 v_2}}\right]$$

$$= \frac{g^2}{N_c^2}(\delta_{\boldsymbol{xu_1}} - \delta_{\boldsymbol{xv_1}})(\delta_{\boldsymbol{yu_2}} - \delta_{\boldsymbol{yv_2}})\mathrm{Tr}(t^a W_{\boldsymbol{u_1}} W^\dagger_{\boldsymbol{v_1}})\mathrm{Tr}(t^b W_{\boldsymbol{u_2}} W^\dagger_{\boldsymbol{v_2}}). \qquad (3.20)$$

Let us consider the action of the first two terms in $h_{ab}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ on this commutator:

$$\frac{g^2}{N_c^2} \int_{\boldsymbol{xyz}} \mathcal{M}_{\boldsymbol{xyz}}(\delta_{\boldsymbol{xu_1}} - \delta_{\boldsymbol{xv_1}})(\delta_{\boldsymbol{yu_2}} - \delta_{\boldsymbol{yv_2}})\left(1 + \widetilde{W}_{\boldsymbol{x}}\widetilde{W}^\dagger_{\boldsymbol{y}}\right)_{ab}\mathrm{Tr}(t^a W_{\boldsymbol{u_1}} W^\dagger_{\boldsymbol{v_1}})\mathrm{Tr}(t^b W_{\boldsymbol{u_2}} W^\dagger_{\boldsymbol{v_2}})$$

$$= \frac{g^2}{N_c} \int_{\boldsymbol{z}} \left(\mathcal{M}_{\boldsymbol{u_1 u_2 z}} - \mathcal{M}_{\boldsymbol{u_1 v_2 z}} - \mathcal{M}_{\boldsymbol{v_1 u_2 z}} + \mathcal{M}_{\boldsymbol{v_1 v_2 z}}\right)\left[\frac{1}{2N_c}\mathrm{tr}(W_{\boldsymbol{u_1}} W^\dagger_{\boldsymbol{v_1}} W_{\boldsymbol{u_1}} W^\dagger_{\boldsymbol{v_2}})\right.$$

$$\left. + \frac{1}{2N_c}\mathrm{tr}(W^\dagger_{\boldsymbol{v_1}} W_{\boldsymbol{u_1}} W^\dagger_{\boldsymbol{v_2}} W_{\boldsymbol{u_2}}) - \frac{1}{N_c}\mathrm{tr}(W_{\boldsymbol{u_1}} W^\dagger_{\boldsymbol{v_1}})\frac{1}{N_c}\mathrm{tr}(W_{\boldsymbol{u_2}} W^\dagger_{\boldsymbol{v_2}})\right].$$

$$(3.21)$$

The r.h.s. of Eq. (3.21) is of order $\bar{\alpha}_s/N_c^2$, and hence it is suppressed by a factor of $1/N_c^2$ compared to the dipole contribution in Eq. (3.18). Similarly, one can verify that the other contributions, due to the last two terms in $h_{ab}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, are also suppressed. Once again, this property has a dual counterpart in the context of the Balitsky–JIMWLK equations: The non–dipolar terms in the evolution equation for the scattering amplitude of a projectile made with *two* dipoles are suppressed at large N_c.

Retaining only the dipolar contribution in Eq. (3.18), we finally obtain the following evolution equation for the weight function in the large–N_c limit:

$$\frac{\partial}{\partial\tau} Z_\tau[\rho] = \frac{\bar{\alpha}_s}{2\pi} \sum_{N=1}^{\infty} \int d\Gamma_N\, P_N(\tau) \sum_{i=1}^{N} \int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{z})$$

$$\times \left[D^\dagger(\boldsymbol{u}_i, \boldsymbol{z})\, D^\dagger(\boldsymbol{z}, \boldsymbol{v}_i) - D^\dagger(\boldsymbol{u}_i, \boldsymbol{v}_i)\right]\prod_{j\neq i} D^\dagger(\boldsymbol{u}_j, \boldsymbol{v}_j)\, \delta[\rho]. \qquad (3.22)$$

As anticipated, this has the same structure as the corresponding equation in the Dipole Model, Eq. (3.7), except for the replacement of the creation operator for a *bare* dipole $D_0^\dagger$ with the corresponding operator for a *dressed* dipole $D^\dagger$. Note that the non–commutativity of the color charge operators ρ^a (cf. the second equation (2.10)) did not play any role in the manipulations leading to Eq. (3.22). This suggests that the ordering of the operators should be irrelevant in the large–N_c limit (at least, for the dipole–related variables). This conclusion will be further supported by the developments in Sect. 4.

From Eq. (3.22), the dipole picture of the target wavefunction can be developed along the same lines as in Ref. [3]. Namely, Eq. (3.22) is consistent with the dipolar structure of the weight function, Eq. (3.10), provided the probability densities P_N which enter the latter

obey the following *Master equation* :

$$\frac{\partial P_N(\boldsymbol{z}_1, ... \boldsymbol{z}_{N-1}; \tau)}{\partial \tau} = -\frac{\bar{\alpha}_s}{2\pi} \left[\sum_{i=1}^{N} \int d^2\boldsymbol{z} \, \mathcal{M}(\boldsymbol{z}_{i-1}, \boldsymbol{z}_i, \boldsymbol{z}) \right] P_N(\boldsymbol{z}_1, ... \boldsymbol{z}_{N-1}; \tau)$$
$$+ \frac{\bar{\alpha}_s}{2\pi} \sum_{i=1}^{N-1} \mathcal{M}(\boldsymbol{z}_{i-1}, \boldsymbol{z}_{i+1}, \boldsymbol{z}_i) \, P_{N-1}(\boldsymbol{z}_1, ..., \boldsymbol{z}_{i-1}, \boldsymbol{z}_{i+1}, ..., \boldsymbol{z}_{N-1}; \tau).$$

(3.23)

The first term in the r.h.s., proportional to P_N, is the *loss term*, which describes the splitting of one dipole from the original configuration of N dipoles. The other terms, proportional to P_{N-1}, are *gain terms* showing the formation of the N–dipole configuration of interest through the splitting of one dipole in original configurations of $N-1$ dipoles.

One can summarize the previous discussion as follows: At large N_c, the Bremsstrahlung Hamiltonian (2.7) acts on the onium weight function built with *dressed* dipoles, Eq. (3.10), in the same way as the Dipole Model Hamiltonian H_{MSW} acts on the weight function (3.1) built with *bare* dipoles. But this does not imply that H_{MSW} can be deduced from H_{BREM} by somehow taking the large–N_c limit of the latter. That is, the derivative expansion of H_{BREM}, Eq. (2.12), up to fourth order in $\delta/\delta\rho$ is neither equivalent to the expression (3.4) of H_{MSW}, nor it reduces to it in some suitable large–N_c limit. In fact, since the expressions for the onium weight function in the Dipole Model, Eq. (3.1), and in the general Bremsstrahlung case, Eq. (3.10), differ from each other by terms of order $(\delta/\delta\rho)^3$ or higher, the respective Hamiltonians *must* differ from each other too (already at order $(\delta/\delta\rho)^3$), in order to yield identical results when acting on different dipole operators. This observation explains, in particular, why the analysis in Ref. [11], which has limited itself to the extraction of the four derivative terms in the Bremsstrahlung Hamiltonian (2.7), has met with difficulties when trying to generate the dipole picture from H_{BREM}.

The above discussion finds its dual counterpart in the analysis of the JIMWLK evolution in Ref. [7] : At large N_c, H_{JIMWLK} acts on the dipole operator $S(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{N_c} \mathrm{tr}(V^\dagger(\boldsymbol{x}) V(\boldsymbol{y}))$, which describes multiple scattering, in the same way as the 'Pomeron' Hamiltonian obtained [7] as the dual partner of the MSW Hamiltonian (3.4) acts on the 'bare' operator $S_0(\boldsymbol{x}, \boldsymbol{y}) = 1 - \frac{g^2}{4N_c}(\alpha_a(\boldsymbol{x}) - \alpha_a(\boldsymbol{y}))^2$, which describes single scattering via two gluon exchange (cf. Eq. (2.1)). The 'Pomeron' Hamiltonian at high density and, respectively, the Dipole Model Hamiltonian (3.4) in the dilute regime are just *effective* Hamiltonians, which generate the correct evolution equations at large N_c (for dipole scattering amplitudes in the first case, and for target dipole densities in the latter), but by acting in a simplified Hilbert space in which each dipole is characterized by the exchange of only two gluons.

It is finally interesting to notice the different ways how the actual QCD Hamiltonians (BREM and JIMWLK) and their effective counterparts (MSW and, respectively, Pomeron) act in the corresponding Hilbert spaces. By construction, H_{MSW} implements the dipole evolution in the most straightforward way: when acting on a collection of (bare) dipoles, it first annihilates one dipole, which is then replaced either by the same dipole, or by a pair of dipoles with one common leg. One may naively expect the general Hamil-

tonian for dipole evolution to have the same structure as H_{MSW}, but in terms of *dressed* (creation and annihilation) operators. However, even at large N_c, the actual dipole evolution in QCD proceeds in a more subtle way: The bilocal operator $\rho_{\boldsymbol{x}}^a \rho_{\boldsymbol{y}}^b$ within H_{BREM} does not annihilate a dipole, rather it rotates this twice, around different directions in color space (see the last line in Eq. (3.15)). These rotations are then compensated by the remaining operators in the Hamiltonian (those involving Wilson lines), which can either restore back the original dipole, or split it into two new dipoles (cf. Eq. (3.18)). Thus, although the initial and final states — prior to, and after the evolution — are colorless dipoles, the evolution proceeds through *colorful* configurations at intermediate steps.

4 Evolution equation for the dipole densities

In the previous section, we have expressed the evolution of the dipole picture in two equivalent ways, which both deal with the ensemble of dipole correlations in the onium: (i) as a renormalization group equation for the color glass weight function $Z_\tau[\rho]$, Eq. (3.22), and (ii) as a Master equation for the dipole probability densities $P_N(\{\boldsymbol{z}_i\}; \tau)$, Eq. (3.23). Alternatively, the same physical evolution can be expressed as an hierarchy of ordinary evolution equations for the dipole k–body densities $n_\tau^{(k)}$ ($k \geq 1$), which in principle can be derived from any of the 'functional' evolutions alluded to above provided one knows the corresponding expressions for the dipole densities. The derivation based on the Master equation (3.23) has been presented in Refs. [4, 16]. In what follows, we shall derive the same equations from the evolution (3.22) of the color glass. This requires to properly identify the operator expressing the dipole number density in the ρ–representation.

For more clarity, let us first recall the *abstract* definition of the dipole number operator, where by 'abstract' we mean an operator which is independent of ρ, and thus of the way in which we measure the dipole distribution. The abstract dipole number density operator reads (for a N–dipole configuration) [3, 4]

$$n_N(\boldsymbol{x}, \boldsymbol{y}; \{\boldsymbol{z}_i\}) = \sum_{i=1}^{N} \delta^{(2)}(\boldsymbol{z}_{i-1} - \boldsymbol{x}) \delta^{(2)}(\boldsymbol{z}_i - \boldsymbol{y}), \tag{4.24}$$

and its expectation value $n_\tau(\boldsymbol{x}, \boldsymbol{y}) \equiv \langle n(\boldsymbol{x}, \boldsymbol{y}) \rangle_\tau$ is obtained as

$$n_\tau(\boldsymbol{x}, \boldsymbol{y}) = \sum_{N=1}^{\infty} \int d\Gamma_N \, P_N(\{\boldsymbol{z}_i\}; \tau) \, n_N(\boldsymbol{x}, \boldsymbol{y}; \{\boldsymbol{z}_i\}). \tag{4.25}$$

By taking a derivative with respect to τ in the above equation and using the Master equation (3.23), one can check that $n_\tau(\boldsymbol{x}, \boldsymbol{y})$ obeys the BFKL equation (2.24). Furthermore, the abstract operator expressing the dipole *pair* density is defined as [4]

$$n_N^{(2)}(\boldsymbol{x}_1, \boldsymbol{y}_1; \boldsymbol{x}_2, \boldsymbol{y}_2) = \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \delta^{(2)}(\boldsymbol{z}_{j-1} - \boldsymbol{x}_1) \delta^{(2)}(\boldsymbol{z}_j - \boldsymbol{y}_1) \delta^{(2)}(\boldsymbol{z}_{k-1} - \boldsymbol{x}_2) \delta^{(2)}(\boldsymbol{z}_k - \boldsymbol{y}_2), \tag{4.26}$$

where the sum is restricted to different pairs of dipoles since we do not want to count the same dipole twice. The equation obeyed by the corresponding expectation value $n_\tau^{(2)}(\boldsymbol{x}_1, \boldsymbol{y}_1; \boldsymbol{x}_2, \boldsymbol{y}_2)$ has been obtained from the Master equation (3.23) in Refs. [4, 16] (see Eq. (5.15) in Ref. [4]), and will be rederived below from Eq. (3.22). Abstract k–body dipole density operators $n_N^{(k)}$ with $k \geq 1$ can be similarly defined [4, 16].

Let us now turn to the representation of these dipole operators in the color glass formalism, where a dipole is described as a source for small–x gluons. Within the Dipole Model, where a dipole can radiate only two gluons, the dipole number densities are built with the 'bare dipole' annihilation operator introduced in Eq. (3.5). Specifically, by using the commutation relation (3.6), one finds (for large N_c and $\boldsymbol{x} \neq \boldsymbol{y}$) :

$$D_0(\boldsymbol{x}, \boldsymbol{y}) \, Z_\tau^{\mathrm{DM}}[\rho] \approx \sum_{N=1}^{\infty} \int d\Gamma_N \, P_N(\tau) \sum_{i=1}^{N} \frac{1}{2} \Big(\delta_{\boldsymbol{z}_{i-1}\boldsymbol{x}} \delta_{\boldsymbol{z}_i\boldsymbol{y}} + \delta_{\boldsymbol{z}_{i-1}\boldsymbol{y}} \delta_{\boldsymbol{z}_i\boldsymbol{x}} \Big) \prod_{j \neq i} D_0^\dagger(\boldsymbol{z}_{j-1}, \boldsymbol{z}_j) \, \delta[\rho].$$
(4.27)

After averaging over ρ, which is tantamount to replacing $D_0^\dagger \to 1$ (since the functional derivatives within $D_0^\dagger$ give zero after integration by parts), this yields:

$$\langle D_0(\boldsymbol{x}, \boldsymbol{y}) \rangle_\tau \equiv \int D[\rho] \, D_0(\boldsymbol{x}, \boldsymbol{y}) \, Z_\tau^{\mathrm{DM}}[\rho]$$
$$\approx \sum_{N=1}^{\infty} \int d\Gamma_N \, P_N(\tau) \sum_{i=1}^{N} \frac{1}{2} \Big(\delta_{\boldsymbol{z}_{i-1}\boldsymbol{x}} \delta_{\boldsymbol{z}_i\boldsymbol{y}} + \delta_{\boldsymbol{z}_{i-1}\boldsymbol{y}} \delta_{\boldsymbol{z}_i\boldsymbol{x}} \Big)$$
$$\equiv (1/2) \Big(n_\tau(\boldsymbol{x}, \boldsymbol{y}) + n_\tau(\boldsymbol{y}, \boldsymbol{x}) \Big),$$
(4.28)

where in writing the last line we have recognized the average dipole number density according to Eq. (4.24). Note that the measure of the dipole density provided by the charge operator D_0 is symmetrized between the quark and antiquark legs of the dipole. One can similarly check that, at large N_c,

$$\langle D_0(\boldsymbol{x_1}, \boldsymbol{y_1}) \, D_0(\boldsymbol{x_2}, \boldsymbol{y_2}) \rangle_\tau = \frac{1}{4} \Big(n_\tau^{(2)}(\boldsymbol{x_1}, \boldsymbol{y_1}; \boldsymbol{x_2}, \boldsymbol{y_2}) + n_\tau^{(2)}(\boldsymbol{y_1}, \boldsymbol{x_1}; \boldsymbol{x_2}, \boldsymbol{y_2})$$
$$+ n_\tau^{(2)}(\boldsymbol{x_1}, \boldsymbol{y_1}; \boldsymbol{y_2}, \boldsymbol{x_2}) + n_\tau^{(2)}(\boldsymbol{y_1}, \boldsymbol{x_1}; \boldsymbol{y_2}, \boldsymbol{x_2}) \Big),$$
(4.29)

so long as we restrict ourselves to *different* dipoles; that is, we exclude configurations such that the two measured dipoles are identical with each other: $\{\boldsymbol{x_1} = \boldsymbol{x_2}; \boldsymbol{y_1} = \boldsymbol{y_2}\}$ or $\{\boldsymbol{x_1} = \boldsymbol{y_2}; \boldsymbol{y_1} = \boldsymbol{x_2}\}$. Besides, we exclude, as usual, the zero–size dipoles; that is, we assume $\boldsymbol{x_1} \neq \boldsymbol{y_1}$ and $\boldsymbol{x_2} \neq \boldsymbol{y_2}$.

We thus see that, within the DM, the (bare) dipole *annihilation* operator plays also the role of a *number* operator. This degeneracy is possible because our present use of creation and annihilation operators is somewhat different from their standard use in quantum mechanics: Whereas the weight function (3.1) plays naturally the role of a "ground state" for the (bare) dipole "Fock space", the color glass expectation value on this 'ground state' involves Z_τ^{DM} itself, and not $|Z_\tau^{\mathrm{DM}}|^2$.

The above identification of D_0 with the dipole number operator is also consistent with the color glass evolution of the DM, as encoded in Eq. (3.7) for Z_τ^{DM}. Using the latter or, more directly, acting with the MSW Hamiltonian (3.4) on the relevant operators built with D_0, one can indeed check that the correlations of D_0 introduced in Eqs. (4.28) and (4.29) obey the evolution equations expected for the respective dipole densities [4, 16]. The most efficient way to perform this calculation is to use the representation of H_{MSW} as a fourth–order differential operator (cf. Eq. (3.2)) acting on functionals of $\rho^a(\boldsymbol{x})$.

Let us now turn to the most interesting case, in which a dipole is allowed to radiate arbitrarily many gluons and the respective creation operator involves Wilson lines, cf. Eq. (3.8). In the previous manipulations leading to the dipole picture, we did not need to introduce the corresponding annihilation operator, because the natural candidate in that respect — the bilocal operator $\rho^a(\boldsymbol{x})\rho^a(\boldsymbol{y})$ — does not enter the Bremsstrahlung Hamiltonian (unlike what happens in the Dipole Model). Yet, as we shall now explain, this operator plays an important role also in the general case, not as an annihilation operator, but rather as a *dipole number operator*. Specifically, if one defines

$$D(\boldsymbol{x}, \boldsymbol{y}) \equiv -\frac{1}{g^2 N_c} \rho_\infty^a(\boldsymbol{x})\rho_\infty^a(\boldsymbol{y}) \quad \text{for} \quad \boldsymbol{x} \neq \boldsymbol{y}, \tag{4.30}$$

which differs from the corresponding DM operator, Eq. (3.5), only by the presence of the time argument $x^+ = \infty$ (that we have temporarily reintroduced for more clarity), then this operator is gauge invariant and obeys the following commutation relation at large N_c

$$[D(\boldsymbol{x}, \boldsymbol{y}), D^\dagger(\boldsymbol{u}, \boldsymbol{v})] \approx \frac{1}{2}\Big(\delta_{\boldsymbol{ux}}\delta_{\boldsymbol{vy}} + \delta_{\boldsymbol{uy}}\delta_{\boldsymbol{vx}}\Big) D^\dagger(\boldsymbol{u}, \boldsymbol{v}), \tag{4.31}$$

which qualifies it as a number operator, as anticipated. To verify the above commutator, note that, for $\boldsymbol{x} \neq \boldsymbol{y}$, Eq. (3.15) yields

$$\begin{aligned}
\left[D_{\boldsymbol{xy}}, D_{\boldsymbol{uv}}^\dagger\right] = {} & -\frac{1}{gN_c^2}(\delta_{\boldsymbol{xu}} - \delta_{\boldsymbol{xv}})\mathrm{tr}(t^a W_{\boldsymbol{u}} W_{\boldsymbol{v}}^\dagger)\,\rho_{\boldsymbol{y}}^a \\
& -\frac{1}{gN_c^2}(\delta_{\boldsymbol{yu}} - \delta_{\boldsymbol{yv}})\mathrm{tr}(t^a W_{\boldsymbol{u}} W_{\boldsymbol{v}}^\dagger)\,\rho_{\boldsymbol{x}}^a \\
& +\frac{1}{2}\Big(\delta_{\boldsymbol{ux}}\delta_{\boldsymbol{vy}} + \delta_{\boldsymbol{uy}}\delta_{\boldsymbol{vx}}\Big) D_{\boldsymbol{uv}}^\dagger,
\end{aligned} \tag{4.32}$$

where only the last term in the r.h.s. survives when acting on the onium weight function (3.10) and at large N_c. (This can be checked through manipulations similar to those in Eqs. (3.19)–(3.21).)

Eq. (4.31) implies that Z_N — the weight function for a given configuration of N dipoles; see Eq. (3.13) — is an eigenstate of D at large N_c (compare to Eq. (4.27)) :

$$D(\boldsymbol{x}, \boldsymbol{y})\, Z_N(\{\boldsymbol{u}_i, \boldsymbol{v}_i\}) \approx \frac{1}{2}\sum_{i=1}^N \Big(\delta_{\boldsymbol{z}_{i-1}\boldsymbol{x}}\delta_{\boldsymbol{z}_i\boldsymbol{y}} + \delta_{\boldsymbol{z}_{i-1}\boldsymbol{y}}\delta_{\boldsymbol{z}_i\boldsymbol{x}}\Big) Z_N(\{\boldsymbol{u}_i, \boldsymbol{v}_i\}), \tag{4.33}$$

which in turn implies (cf. Eq. (4.25)) :

$$\langle D(\boldsymbol{x}, \boldsymbol{y}) \rangle_\tau \equiv \int D[\rho]\, D(\boldsymbol{x}, \boldsymbol{y})\, Z_\tau[\rho] \approx \frac{1}{2}\Big(n_\tau(\boldsymbol{x}, \boldsymbol{y}) + n_\tau(\boldsymbol{y}, \boldsymbol{x}) \Big). \qquad (4.34)$$

More generally, when acting on $Z_\tau[\rho]$ with a string of k D–operators corresponding to non–identical dipoles, then one measures the k–body dipole density (symmetrized under the exchange of the quark and antiquark legs of each dipole); e.g., for $k = 2$, one finds the analog of Eq. (4.28). Note that, although two operators like $D(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $D(\boldsymbol{x_2}, \boldsymbol{y_2})$ do not commute with each other, this non–commutativity is irrelevant when computing their action on the onium weight function and for large N_c.

In the remaining part of this section, we shall verify that the above interpretation of the operator $D(\boldsymbol{x}, \boldsymbol{y})$, Eq. (4.30), as the dipole number operator is consistent with the evolution of the correlations of ρ generated by the Bremsstrahlung Hamiltonian (2.7) at large N_c. In Sect. 2, we have already performed this check for the 2–point function: when acting on $\rho^a(\boldsymbol{x})\rho^a(\boldsymbol{y})$, H_{BREM} generates the BFKL equation (2.24) for $\langle D(\boldsymbol{x}, \boldsymbol{y}) \rangle_\tau$, in agreement with Eq. (4.34). In what follows, we shall use the results of Sect. 3 to perform the corresponding check for the dipole pair density. That is, we shall verify that the evolution equation satisfied at large N_c by the following 4–point correlation function:

$$\langle D(\boldsymbol{x_1}, \boldsymbol{y_1}) D(\boldsymbol{x_2}, \boldsymbol{y_2}) \rangle_\tau = \frac{1}{g^4 N_c^2} \langle \rho^a(\boldsymbol{x_1}) \rho^a(\boldsymbol{y_1}) \rho^b(\boldsymbol{x_2}) \rho^b(\boldsymbol{y_2}) \rangle_\tau, \qquad (4.35)$$

is consistent with the known evolution equation for $n^{(2)}(\boldsymbol{x_1}, \boldsymbol{y_1}; \boldsymbol{x_2}, \boldsymbol{y_2})$, including the interesting, 'fluctuation', term (i.e., the term linear in n which describes the formation of the dipole pair $\{(\boldsymbol{x_1}, \boldsymbol{y_1}), (\boldsymbol{x_2}, \boldsymbol{y_2})\}$ through the splitting of one dipole in the last step of the evolution) [4, 16].

Using Eq. (3.22), we obtain

$$\frac{\partial}{\partial \tau} \langle D(\boldsymbol{x_1}, \boldsymbol{y_1}) D(\boldsymbol{x_2}, \boldsymbol{y_2}) \rangle_\tau = \frac{\bar{\alpha}_s}{2\pi} \sum_{N=1}^{\infty} \int d\Gamma_N\, P_N(\tau) \sum_{i=1}^{N} \int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{u}_i \boldsymbol{v}_i \boldsymbol{z}} \int D[\rho]\, D(\boldsymbol{x_1}, \boldsymbol{y_1}) D(\boldsymbol{x_2}, \boldsymbol{y_2})$$
$$\times \Big[D^\dagger(\boldsymbol{u}_i, \boldsymbol{z})\, D^\dagger(\boldsymbol{z}, \boldsymbol{v}_i) - D^\dagger(\boldsymbol{u}_i, \boldsymbol{v}_i) \Big] \prod_{j \neq i} D^\dagger(\boldsymbol{u}_j, \boldsymbol{v}_j)\, \delta[\rho]\,.$$
$$(4.36)$$

To evaluate the r.h.s. of this equation, we need to move the operators DD all the way to the right using the commutator Eq. (4.31) and then set $D^\dagger = 1$. Clearly, the following commutator vanishes

$$\big[\, D_{\boldsymbol{x_1 y_1}}, \, [\, D_{\boldsymbol{x_2 y_2}}, \, D^\dagger_{\boldsymbol{u_k v_k}} \,] \,\big] \qquad (\text{any } k), \qquad (4.37)$$

unless the two external dipoles completely overlap, an uninteresting situation that we exclude. Moreover, terms like

$$\big[\, D_{\boldsymbol{x_1 y_1}}, D^\dagger_{\boldsymbol{u_j v_j}} \big]\big[\, D_{\boldsymbol{x_2 y_2}}, D^\dagger_{\boldsymbol{u_k v_k}} \big] \qquad (j, k \neq i), \qquad (4.38)$$

do not contribute either, because the corresponding coefficient vanishes after setting $D^\dagger = 1$. Therefore, we can replace the operator part in Eq. (4.36) with

$$D_{\boldsymbol{x}_1\boldsymbol{y}_1} D_{\boldsymbol{x}_2\boldsymbol{y}_2}(D^\dagger_{\boldsymbol{u}_i\boldsymbol{z}} D^\dagger_{\boldsymbol{z}\boldsymbol{v}_i} - D^\dagger_{\boldsymbol{u}_i\boldsymbol{v}_i})\prod_{j\neq i} D^\dagger_{\boldsymbol{u}_j\boldsymbol{v}_j}$$

$$\rightarrow \big[\, D_{\boldsymbol{x}_1\boldsymbol{y}_1},\, (D^\dagger_{\boldsymbol{u}_i\boldsymbol{z}} D^\dagger_{\boldsymbol{z}\boldsymbol{v}_i} - D^\dagger_{\boldsymbol{u}_i\boldsymbol{v}_i})\,\big]\Big(\sum_{j\neq i}[\, D_{\boldsymbol{x}_2\boldsymbol{y}_2},\, D^\dagger_{\boldsymbol{u}_j\boldsymbol{v}_j}\,]\Big) + \{1\leftrightarrow 2\}$$

$$+ [\, D_{\boldsymbol{x}_1\boldsymbol{y}_1},\, D^\dagger_{\boldsymbol{u}_i\boldsymbol{z}}\,]\,[\, D_{\boldsymbol{x}_2\boldsymbol{y}_2},\, D^\dagger_{\boldsymbol{z}\boldsymbol{v}_i}\,] + \{1\leftrightarrow 2\}. \qquad (4.39)$$

The second line of Eq. (4.39) describes the BFKL evolution of one of the two dipoles in the pair (and for large N_c). To see this, note that, e.g.,

$$\sum_i \int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{u}_i\boldsymbol{v}_i\boldsymbol{z}}\big[\, D_{\boldsymbol{x}_1\boldsymbol{y}_1},\, (D^\dagger_{\boldsymbol{u}_i\boldsymbol{z}} D^\dagger_{\boldsymbol{z}\boldsymbol{v}_i} - D^\dagger_{\boldsymbol{u}_i\boldsymbol{v}_i})\,\big]\Big|_{D^\dagger=1}$$

$$= \frac{1}{2}\sum_i\Big\{\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{v}_i\boldsymbol{y}_1}\delta_{\boldsymbol{x}_1\boldsymbol{u}_i} + \mathcal{M}_{\boldsymbol{u}_i\boldsymbol{y}_1\boldsymbol{x}_1}\delta_{\boldsymbol{y}_1\boldsymbol{v}_i} - \int_{\boldsymbol{z}}\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{y}_1\boldsymbol{z}}\delta_{\boldsymbol{x}_1\boldsymbol{u}_i}\delta_{\boldsymbol{y}_1\boldsymbol{v}_i} + \{\boldsymbol{x}_1\leftrightarrow\boldsymbol{y}_1\}\Big\}$$

$$= \frac{1}{2}\sum_i\int_{\boldsymbol{z}}\Big\{\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{z}\boldsymbol{y}_1}\delta_{\boldsymbol{x}_1\boldsymbol{u}_i}\delta_{\boldsymbol{z}\boldsymbol{v}_i} + \mathcal{M}_{\boldsymbol{z}\boldsymbol{y}_1\boldsymbol{x}_1}\delta_{\boldsymbol{z}\boldsymbol{u}_i}\delta_{\boldsymbol{y}_1\boldsymbol{v}_i} - \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{y}_1\boldsymbol{z}}\delta_{\boldsymbol{x}_1\boldsymbol{u}_i}\delta_{\boldsymbol{y}_1\boldsymbol{v}_i} + \{\boldsymbol{x}_1\leftrightarrow\boldsymbol{y}_1\}\Big\},$$

$$(4.40)$$

which represents the BFKL evolution of the dipole $(\boldsymbol{x}_1,\boldsymbol{y}_1)$, cf. Eq. (2.24).

The last line of Eq. (4.39) is the fluctuation term that we are primarily interested in. For large N_c, the commutators there can be evaluated according to Eq. (4.31). Note that the use of Eq. (4.31) (instead of the exact relation (4.32)) automatically avoids contributions in which the external dipoles get mixed with each other under the action of the commutators (that is, contributions where the two factors of ρ from a same external dipole get contracted with factors of W coming from different internal dipoles); this was to be expected, since such contributions are indeed suppressed at large N_c.

After also integrating over $\boldsymbol{z}$ and summing over N, we finally obtain

$$\frac{\partial}{\partial\tau}\langle D_{\boldsymbol{x}_1,\boldsymbol{y}_1} D_{\boldsymbol{x}_2,\boldsymbol{y}_2}\rangle_\tau = \big[H^{(1)}_{\mathrm{BFKL}} + H^{(2)}_{\mathrm{BFKL}}\big]\langle D_{\boldsymbol{x}_1,\boldsymbol{y}_1} D_{\boldsymbol{x}_2,\boldsymbol{y}_2}\rangle_\tau$$

$$+ \frac{\bar{\alpha}_s}{4\pi}\Big\{\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{y}_2\boldsymbol{x}_2}\delta_{\boldsymbol{x}_2\boldsymbol{y}_1}\langle D_{\boldsymbol{x}_1\boldsymbol{y}_2}\rangle_\tau + \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{y}_1}\delta_{\boldsymbol{y}_1\boldsymbol{y}_2}\langle D_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_\tau$$

$$+ \mathcal{M}_{\boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{x}_1}\delta_{\boldsymbol{x}_1\boldsymbol{x}_2}\langle D_{\boldsymbol{y}_1\boldsymbol{y}_2}\rangle_\tau + \mathcal{M}_{\boldsymbol{y}_1\boldsymbol{x}_2\boldsymbol{x}_1}\delta_{\boldsymbol{x}_1\boldsymbol{y}_2}\langle D_{\boldsymbol{y}_1\boldsymbol{x}_2}\rangle_\tau\Big\}, \qquad (4.41)$$

with the compact notation

$$H^{(1)}_{\mathrm{BFKL}}\langle D_{\boldsymbol{x}_1,\boldsymbol{y}_1} D_{\boldsymbol{x}_2,\boldsymbol{y}_2}\rangle_\tau \equiv \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}}\Big[-\mathcal{M}(\boldsymbol{x}_1,\boldsymbol{y}_1,\boldsymbol{z})\,\langle D_{\boldsymbol{x}_1,\boldsymbol{y}_1} D_{\boldsymbol{x}_2,\boldsymbol{y}_2}\rangle_\tau$$

$$+ \mathcal{M}(\boldsymbol{x}_1,\boldsymbol{z},\boldsymbol{y}_1)\,\langle D_{\boldsymbol{x}_1,\boldsymbol{z}} D_{\boldsymbol{x}_2,\boldsymbol{y}_2}\rangle_\tau + \mathcal{M}(\boldsymbol{z},\boldsymbol{y}_1,\boldsymbol{x}_1)\,\langle D_{\boldsymbol{z},\boldsymbol{y}_1} D_{\boldsymbol{x}_2,\boldsymbol{y}_2}\rangle_\tau\Big]. \qquad (4.42)$$

In identifying the expectation values in the r.h.s.'s of the above equations, we have used the (dressed–dipole version of the) relations (4.28) and (4.29). The fluctuation terms are

the terms proportional to the average dipole number density $\langle D \rangle_\tau$ in Eq. (4.41). Using Eq. (4.28) and (4.29) once again, one can check that Eq. (4.41) is equivalent to Eq. (5.15) of Ref. [4]. This confirms the interpretation of $\langle DD \rangle_\tau$ as the average dipole pair density.

5 Bremsstrahlung effects on dipole–onium scattering

So far, we have almost exclusively focused on the evolution of the target wavefunction : the large–N_c evolution in the dilute regime has been formulated either as an evolution equation for the onium weight function, Eq. (3.22), or as a set of coupled evolution equations for those correlations of ρ which have the meaning of dipole densities (see, e.g., Eqs. (2.24) and (4.41)). But the scattering amplitudes for external projectiles involve also different correlations of ρ (or, more precisely, of $\alpha \equiv \alpha_\infty^a(\boldsymbol{x})$; see, e.g., Eq. (2.5)), so it would be interesting to establish the corresponding evolution equations as well, at least at large N_c. As anticipated in the Introduction, the evolution equations for scattering amplitudes generated by H_{BREM} are *a priori* more general than those previously derived in the literature from the dipole picture [4–6], in the sense of including additional processes in which individual target dipoles exchange more than two gluons with the projectile. However, our main purpose in this section is not to derive such more general equations, but rather to give an argument, based on an explicit example, that the additional effects are in fact suppressed at sufficiently high energy.

The simplest scattering problem which is sensitive to Bremsstrahlung (i.e., to gluon number fluctuations) in the target wavefunction is the scattering with two external dipoles. (For a single external dipole, H_{BREM} generates the standard BFKL equation; see below.) To lowest order in perturbation theory, each external dipole can exchange two gluons with the target, cf. Eq. (2.1). Therefore, the leading–order contribution to the amplitude for the *simultaneous* scattering of the incoming dipoles reads

$$\langle T^{(2)}(\boldsymbol{x}_1, \boldsymbol{y}_1; \boldsymbol{x}_2, \boldsymbol{y}_2) \rangle_\tau \simeq \frac{g^4}{16 N_c^2} \left\langle (\alpha_a(\boldsymbol{x}_1) - \alpha_a(\boldsymbol{y}_1))^2 (\alpha_b(\boldsymbol{x}_2) - \alpha_b(\boldsymbol{y}_2))^2 \right\rangle_\tau, \qquad (5.1)$$

and involves a total exchange of four gluons (two with each external dipole). These four gluons can be absorbed either by two different dipoles in the target wavefunction, or by a single such a dipole, and the two type of processes are parametrically of the same order in α_s and $1/N_c$. In particular, they both contribute in the large–N_c limit. Still, as we shall argue below, the corresponding contributions behave differently when increasing the energy, in such a way that multiple exchanges with the same target dipole are relatively suppressed at high energy.

To see this, we need to evaluate the average in Eq. (5.1) with the onium weight function (3.10). The scattering operator for a single (external) dipole, that is,

$$T_0(\boldsymbol{x}, \boldsymbol{y}) \equiv \frac{g^2}{4 N_c} \left(\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{y}) \right)^2, \qquad (5.2)$$

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can be conveniently expressed in terms of the (target) dipole number operator introduced in Sect. 4. Namely, after inverting the Poisson equation (2.4) to relate α^a to ρ^a, one obtains

$$\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{y}) = \int_{\boldsymbol{u}} \mathcal{G}(\boldsymbol{u}|\boldsymbol{x},\boldsymbol{y})\,\rho^a(\boldsymbol{u})\,, \qquad \mathcal{G}(\boldsymbol{u}|\boldsymbol{x},\boldsymbol{y}) \equiv \frac{1}{4\pi}\ln\frac{(\boldsymbol{u}-\boldsymbol{y})^2}{(\boldsymbol{u}-\boldsymbol{x})^2}\,, \qquad (5.3)$$

which then allows us to successively write

$$\begin{aligned}
T_0(\boldsymbol{x},\boldsymbol{y}) &= \frac{g^2}{4N_c} \int_{\boldsymbol{u},\boldsymbol{v}} \mathcal{G}(\boldsymbol{u}|\boldsymbol{x},\boldsymbol{y})\mathcal{G}(\boldsymbol{v}|\boldsymbol{x},\boldsymbol{y})\,\rho^a(\boldsymbol{u})\rho^a(\boldsymbol{v}) \\
&= -\frac{g^2}{8N_c} \int_{\boldsymbol{u},\boldsymbol{v}} \big[\mathcal{G}(\boldsymbol{u}|\boldsymbol{x},\boldsymbol{y}) - \mathcal{G}(\boldsymbol{v}|\boldsymbol{x},\boldsymbol{y})\big]^2 \rho^a(\boldsymbol{u})\rho^a(\boldsymbol{v}) \\
&= \int_{\boldsymbol{u},\boldsymbol{v}} \mathcal{A}_0(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{u},\boldsymbol{v})\,D(\boldsymbol{u},\boldsymbol{v}), \qquad\qquad\qquad\qquad (5.4)
\end{aligned}$$

where in going from the first to the second line we have used the fact that the system is globally color neutral : $\int_{\boldsymbol{u}} \rho^a(\boldsymbol{u}) = 0$. The operator $D(\boldsymbol{u},\boldsymbol{v})$ has been defined in Eq. (4.30), and $\mathcal{A}_0$ is the amplitude for dipole–dipole scattering in the two–gluon exchange approximation and for large N_c :

$$\mathcal{A}_0(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{u},\boldsymbol{v}) = \frac{\alpha_s^2}{8}\left[\ln\frac{(\boldsymbol{x}-\boldsymbol{v})^2(\boldsymbol{y}-\boldsymbol{u})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{v})^2}\right]^2. \qquad (5.5)$$

It is now straightforward to compute the average scattering amplitude for a single external dipole: using Eq. (4.33) together with the symmetry of the dipole–dipole amplitude $\mathcal{A}_0(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{u},\boldsymbol{v})$ under the exchange $\boldsymbol{u} \leftrightarrow \boldsymbol{v}$, one immediately obtains

$$\langle T(\boldsymbol{x},\boldsymbol{y})\rangle_\tau = \int_{\boldsymbol{u},\boldsymbol{v}} \mathcal{A}_0(\boldsymbol{x},\boldsymbol{y}|\boldsymbol{u},\boldsymbol{v})\,n_\tau(\boldsymbol{u},\boldsymbol{v}), \qquad (5.6)$$

which is the relation expected within the dipole picture [1–3]. By using this relation together with Eq. (2.24) for n_τ, one can show [3] that $\langle T(\boldsymbol{x},\boldsymbol{y})\rangle_\tau$ obeys the standard BFKL equation, as anticipated.

To similarly evaluate Eq. (5.1), we shall use Eq. (5.4) for both external dipoles. We have

$$\langle T^{(2)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\boldsymbol{x}_2,\boldsymbol{y}_2)\rangle_\tau = \int_{\boldsymbol{u}_i,\boldsymbol{v}_i} \mathcal{A}_0(\boldsymbol{x}_1,\boldsymbol{y}_1|\boldsymbol{u}_1,\boldsymbol{v}_1)\,\mathcal{A}_0(\boldsymbol{x}_2,\boldsymbol{y}_2|\boldsymbol{u}_2,\boldsymbol{v}_2)\,\langle D(\boldsymbol{u}_1,\boldsymbol{v}_1)\,D(\boldsymbol{u}_2,\boldsymbol{v}_2)\rangle_\tau,$$

$$(5.7)$$

where after using again Eq. (4.33) and the symmetries of the tree–level amplitude $\mathcal{A}_0$, one can effectively replace (at large N_c)

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$$\langle D(\boldsymbol{u}_1, \boldsymbol{v}_1) \, D(\boldsymbol{u}_2, \boldsymbol{v}_2)\rangle_\tau \to \Big\langle \sum_{i=1}^{N} \delta^{(2)}(\boldsymbol{z}_{i-1} - \boldsymbol{u}_1)\delta^{(2)}(\boldsymbol{z}_i - \boldsymbol{v}_1) \sum_{k=1}^{N} \delta^{(2)}(\boldsymbol{z}_{k-1} - \boldsymbol{u}_2)\delta^{(2)}(\boldsymbol{z}_k - \boldsymbol{v}_2)\Big\rangle_\tau$$
$$= n_\tau^{(2)}(\boldsymbol{u}_1, \boldsymbol{v}_1; \boldsymbol{u}_2, \boldsymbol{v}_2) + \delta^{(2)}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\delta^{(2)}(\boldsymbol{v}_1 - \boldsymbol{v}_2)n_\tau(\boldsymbol{u}_1, \boldsymbol{v}_1).$$
(5.8)

We have recognized here the average dipole density in the target n_τ and also the average dipole pair density $n_\tau^{(2)}$ according to their definitions in Eqs. (4.24)–(4.26). When the above expression is inserted in the r.h.s. of Eq. (5.7), we finally obtain:

$$\langle T^{(2)}(\boldsymbol{x}_1, \boldsymbol{y}_1; \boldsymbol{x}_2, \boldsymbol{y}_2)\rangle_\tau = \int_{\boldsymbol{u}_i, \boldsymbol{v}_i} \mathcal{A}_0(\boldsymbol{x}_1, \boldsymbol{y}_1 | \boldsymbol{u}_1, \boldsymbol{v}_1) \, \mathcal{A}_0(\boldsymbol{x}_2, \boldsymbol{y}_2 | \boldsymbol{u}_2, \boldsymbol{v}_2) \, n_\tau^{(2)}(\boldsymbol{u}_1, \boldsymbol{v}_1; \boldsymbol{u}_2, \boldsymbol{v}_2)$$
$$+ \int_{\boldsymbol{u}, \boldsymbol{v}} \mathcal{A}_0(\boldsymbol{x}_1, \boldsymbol{y}_1 | \boldsymbol{u}, \boldsymbol{v}) \, \mathcal{A}_0(\boldsymbol{x}_2, \boldsymbol{y}_2 | \boldsymbol{u}, \boldsymbol{v}) \, n_\tau(\boldsymbol{u}, \boldsymbol{v}),$$
(5.9)

with a clear physical interpretation for the two terms in the r.h.s. : The first term, proportional to $n_\tau^{(2)}$, describes the scattering with two *different* dipoles in the target, while the second term, proportional to n_τ, represents a four–gluon exchange with a *single* dipole. Incidentally, the above derivation of Eq. (5.9) confirms that no ordering ambiguities appear in the evaluation of the scattering amplitudes at large N_c.

Previous applications of the dipole picture to scattering [1–4, 6] were all based on the assumption that a dipole can exchange only two gluons, and thus they have ignored the second term in Eq. (5.9). Using only the first term there, together with the known equation for $n_\tau^{(2)}$ (essentially, Eq. (4.41)), one has derived an evolution equation for $\langle T^{(2)}\rangle_\tau$ [4, 6] which includes the essential 'fluctuation term' through which $\langle T^{(2)}\rangle_\tau$ gets built from $\langle T\rangle_\tau$ in the dilute regime. (This is induced by the fluctuation terms in Eq. (4.41).) In principle, one can similarly use Eq. (5.9) together with the known equations for $n_\tau^{(2)}$ and n_τ to deduce the most general evolution equation for $\langle T^{(2)}\rangle_\tau$ in the dilute regime and at large N_c. In practice, this might be however tedious, as it requires to invert Eqs. (5.6) and (5.9) in order to express $n_\tau^{(2)}$ and n_τ in terms of the scattering amplitudes.

But even without any detailed calculation, it is clear by inspection of Eq. (5.9) that at high energy (while still in the dilute regime, though, for the dipole picture to apply), the contribution involving the dipole pair density $n_\tau^{(2)}$ is in fact the dominant one: Indeed, with increasing energy, n_τ grows like a BFKL pomeron, $n_\tau \sim \exp\{\omega \bar{\alpha}_s \tau\}$, whereas $n_\tau^{(2)}$ grows like a Pomeron squared: $n_\tau^{(2)} \sim \exp\{2\omega \bar{\alpha}_s \tau\}$ (ω is a pure number). Therefore, the contribution proportional to $n_\tau^{(2)}$ dominates as soon as $\bar{\alpha}_s \tau \gtrsim 1$, which is the interesting regime at high energy[5]. This implies not only that, for $\bar{\alpha}_s \tau \gtrsim 1$, one can neglect n_τ next to $n_\tau^{(2)}$ in the r.h.s. of Eq. (5.9), but also that, in writing an equation for $\langle T^{(2)}\rangle_\tau$ which should describe its evolution from the low energy regime at $\tau \sim 0$ up to the high energy where $\bar{\alpha}_s \tau \gtrsim 1$, it is enough to keep trace of the terms coming from the evolution of $n_\tau^{(2)}$. This is what has been done in Ref. [4, 6].

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[5] This condition leaves a parametrically large window for the applicability of the dipole picture, which ceases to be valid when $n_\tau \sim 1/\alpha_s^2$ or $\bar{\alpha}_s \tau \sim \ln(1/\alpha_s^2)$ [1, 4].

Although certainly correct (in view of the previous arguments), the last statement is nevertheless quite subtle, as it can be appreciated when trying to identify the fluctuation term (the contribution proportional to $\langle T \rangle_\tau$) in the evolution equation for $\langle T^{(2)} \rangle_\tau$ derived from Eq. (5.9). By taking a derivative w.r.t. τ in Eq. (5.9) and using the equations for $n_\tau^{(2)}$ and n_τ, one finds two type of contributions which are proportional to n_τ (and thus to $\langle T \rangle_\tau$, cf. Eq. (5.6)) : the 'genuine' fluctuation term coming from the evolution (4.41) of $n_\tau^{(2)}$, and the terms describing the BFKL evolution (2.24) of n_τ. Both types of contributions are parametrically of the same order — namely, of order $\bar{\alpha}_s \alpha_s^2 \langle T \rangle_\tau$—, and thus contribute on equal footing to the growth of $\langle T^{(2)} \rangle_\tau$ in the dilute regime. This seems to contradict the results in Ref. [4, 6], where only the 'genuine' fluctuation term has been included.

However, this contradiction is only illusory: First, to properly identify the fluctuation terms, one need to reexpress everywhere $n_\tau^{(2)}$ in terms of $\langle T^{(2)} \rangle_\tau$, according to Eq. (5.9); this operation modifies the fluctuation terms since, schematically, $\mathcal{A}_0^2 n_\tau^{(2)} = \langle T^{(2)} \rangle_\tau - \mathcal{A}_0^2 n_\tau$. Second, the additional fluctuation terms (besides the 'genuine' one) which persists after the previous operation have only the role to compensate the spurious double–Pomeron contribution which emerges from the evolution of that piece of the initial condition $\langle T^{(2)} \rangle_0$ which is introduced by the second term in Eq. (5.9). Therefore, if one starts with the following initial conditions at $\tau = 0$ (i.e., a target made with a single dipole):

$$n_0(\boldsymbol{u}, \boldsymbol{v}) = \delta^{(2)}(\boldsymbol{u} - \boldsymbol{u}_0)\delta^{(2)}(\boldsymbol{v} - \boldsymbol{v}_0), \qquad n_0^{(2)} = 0 \,, \qquad (5.10)$$

which in turn implies

$$\langle T^{(2)}(\boldsymbol{x}_1, \boldsymbol{y}_1; \boldsymbol{x}_2, \boldsymbol{y}_2) \rangle_0 = \mathcal{A}_0(\boldsymbol{x}_1, \boldsymbol{y}_1|\boldsymbol{u}_0, \boldsymbol{v}_0) \, \mathcal{A}_0(\boldsymbol{x}_2, \boldsymbol{y}_2|\boldsymbol{u}_0, \boldsymbol{v}_0) \,, \qquad (5.11)$$

then the dominant contribution to $\langle T^{(2)} \rangle_\tau$ at high energy — the one which grows like a double–Pomeron, $\sim \exp\{2\omega\bar{\alpha}_s\tau\}$ — comes entirely from the 'genuine' fluctuation term, and *not* from the evolution of the above initial condition. The latter will give only a subleading contribution, $\sim \exp\{\omega\bar{\alpha}_s\tau\}$.

These considerations can be clarified with the help of a simple example, borrowed from Ref. [4], which has the advantage to be easily solvable while keeping the non–trivial features of interest here[6]. Namely, let us replace the dipoles par point–like particles which live at a fixed point (so there is no spatial dimension involved in their dynamics), and whose number distribution evolves according to the following hierarchy of equations:

$$\frac{dn_\tau}{d\tau} = \alpha \, n_\tau, \qquad \frac{dn_\tau^{(2)}}{d\tau} = 2\alpha \left[n_\tau^{(2)} + n_\tau \right], \quad \dots \qquad (5.12)$$

which mimic the equations satisfied by the dipole densities in the dipole picture. The

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[6] It is likely that the following argument is similar to that developed in a different context by Braun and Vacca [24], but we have not been able to clearly establish the correspondence between the two problems. See also the discussion in Ref. [6].

solution corresponding to the initial conditions $n(0) = n_0$ and $n^{(2)}(0) = 0$ reads:

$$n_\tau = n_0 \exp(\alpha\tau),$$
$$n_\tau^{(2)} = 2n_0 \exp(2\alpha\tau) - 2n_0 \exp(\alpha\tau), \; \ldots \tag{5.13}$$

where $n_\tau^{(2)}$ has been generated by the term linear in n_τ in the r.h.s. of the second equation (5.12); this is the analog of the 'genuine fluctuation term' in the present model, and plays the role of a source for $n_\tau^{(2)}$ (so like the actual fluctuation terms in Eq. (4.41)). If one further introduces the analog of the 'scattering amplitudes' :

$$T_\tau = n_\tau, \qquad T_\tau^{(2)} = An_\tau^{(2)} + Bn_\tau, \tag{5.14}$$

then clearly

$$T_\tau^{(2)} = 2An_0 \left[\exp(2\alpha\tau) - \exp(\alpha\tau)\right] + Bn_0 \exp(\alpha\tau) \approx 2An_0 \exp(2\alpha\tau), \tag{5.15}$$

where the approximate equality holds at large time ($\alpha\tau \gg 1$), and the corresponding contribution to $T_\tau^{(2)}$ is entirely coming from $n_\tau^{(2)}$ (like in Eq. (5.9)). Let us now construct the evolution equation for $T_\tau^{(2)}$ and follow the fluctuation terms:

$$\begin{aligned}
\frac{dT_\tau^{(2)}}{d\tau} &= A\frac{dn_\tau^{(2)}}{d\tau} + B\frac{dn_\tau}{d\tau} = 2A\alpha \left[n_\tau^{(2)} + n_\tau\right] + B\alpha\, n_\tau \\
&= 2\alpha\, T_\tau^{(2)} + (2A - B)\alpha\, T_\tau,
\end{aligned} \tag{5.16}$$

where the second line identifies $(2A - B)\alpha\, T_\tau$ as the fluctuation term. This involves the two types of contributions alluded to before: $2A\alpha\, T_\tau$ is the genuine fluctuation term introduced by the evolution of $n_\tau^{(2)}$, whereas $(-B)\alpha\, T_\tau$ has been induced by the BFKL evolution of n_τ. With $T_\tau = n_0 \exp(\alpha\tau)$, the above equation is solved by

$$T_\tau^{(2)} = T_0^{(2)} \exp(2\alpha\tau) + (2A - B)n_0 \left[\exp(2\alpha\tau) - \exp(\alpha\tau)\right], \tag{5.17}$$

where both types of fluctuations seem to contribute to the dominant, 'double–Pomeron', rise at large time. However, by recalling that $T_0^{(2)} = Bn_0$, cf. Eq. (5.14), one immediately sees that the 'double–Pomeron' terms proportional to B do actually cancel between the contribution of the initial condition and that of the fluctuation terms. The remaining 'double–Pomeron' term, proportional to A, is the one generated by the genuine fluctuation term, so like in the direct calculation leading to Eq. (5.15). Thus, the same dominant behaviour at large time would have been obtained by solving the simplified equation:

$$\frac{dT_\tau^{(2)}}{d\tau} = 2\alpha\, T_\tau^{(2)} + 2A\alpha\, T_\tau, \tag{5.18}$$

(which arises by assuming that $T_\tau^{(2)} = An_\tau^{(2)}$) together with the initial condition $T_0^{(2)} = 0$. Eq. (5.18) is the analog of the equation for $\langle T^{(2)}\rangle_\tau$ derived in Refs. [4, 6], whereas Eq. (5.16) corresponds to the more complete equation that would be obtained in QCD at large N_c after including the effects of Bremsstrahlung in the target wavefunction.

The simple example above also emphasizes the importance of correctly adjusting the initial condition to the approximations that we perform on the evolution equation: If, as in Refs. [4, 6], we restrict ourselves to the two–gluon exchange approximation in the construction of the evolution equation for $\langle T^{(2)} \rangle_\tau$, then the same approximation must be performed also on the initial condition. For instance, if at $\tau = 0$ the target is a bare dipole, cf. Eq. (5.10), then the initial condition to be used for the equations in Refs. [6] is $\langle T^{(2)} \rangle_0 = 0$, and not Eq. (5.11). This amounts to consistently neglect contributions like the second term in Eq. (5.9) at all places.

6 Conclusions

In this paper we have shown that the recently developed theory for Bremsstrahlung in the QCD evolution with increasing energy [8–10] is consistent with the dipole picture in the large–N_c limit. The characteristic feature of this theory is that the quark and the antiquark parts of a color dipole are represented as color sources which can radiate arbitrarily many gluons with relatively small longitudinal momenta. This generalizes previous 'color glass' descriptions of the onium wavefunction, in which the individual dipoles were allowed to radiate only two gluons [3–6]. The energy evolution of the ensemble of color sources is known for arbitrary N_c — this is described by a two–dimensional, Hamiltonian, field theory with SU(N_c)–like Poisson brackets —, and our objective in this paper has been to demonstrate that, for large N_c, this evolution can be reformulated in terms of color dipoles which evolve through dipole splitting. To that aim, we have identified the operator which describes, within the Hamiltonian theory, a quark–antiquark dipole 'dressed' by the radiation, and then we have shown that, at large N_c, the action of the Hamiltonian on a collection of such dipoles consists in the splitting of any of the original dipoles into two new dipoles with one common leg. This confirms the fact that the effective degrees of freedom for high energy evolution in QCD at large N_c and in the dilute regime are quark–antiquark color dipoles, as originally demonstrated at an abstract level (i.e., without specifying the way how the dipoles are actually measured) in the pioneering papers by Mueller [1, 2].

Whereas the emergence of the dipole picture at large N_c was to be expected (in view of the general results in Ref. [1]), the technical manipulations necessary to demonstrate it turned out to be quite complex. In particular, we have found that the *structure* of the Hamiltonian cannot be further simplified when going to the large–N_c limit. Rather, it is its *action* on the 'onium weight function' (built with specific dipole creation operators) which reduces to dipole splitting at large N_c. Thus the present construction provides a rather subtle generalization of previous formalisms based on the two–gluon exchange approximation [3, 5]: Whereas the structure of the wavefunction in terms of dipole creation operators is formally the same (with different meanings for the creation operators though), the corresponding Hamiltonians are very different, and cannot be directly related to each other via some large–N_c approximations.

This situation is reminiscent of that encountered in a recent large–N_c analysis of the

JIMWLK evolution in the high density regime [7], and in fact the duality between the two Hamiltonian field theories (JIMWLK and Bremsstrahlung) [9, 10] has played an important role in the present analysis. Namely, the action of the JIMWLK Hamiltonian on scattering operators for external dipoles turns out to be dual to that of the Bremsstrahlung Hamiltonian on creation operators for internal dipoles. Because of that, many of the technical manipulations in this paper are similar (more properly, dual) to those encountered in the derivation of the Balitsky equations [22] from the JIMWLK equation [13, 20].

We have finally investigated the effects of Bremsstrahlung on the scattering between two external dipoles and a dilute onium. We have shown that, although some new effects appear, as associated with the double scattering off a same target dipole, such effects are in fact suppressed at high energy as compared to those already included in the two–gluon exchange approximation. This conclusion, which is corroborated by the recent analysis in Ref. [18], implies that the evolution equations with 'pomeron loops' derived in Refs. [4–6] are indeed the correct equations in QCD at large N_c and for sufficiently high energy.

It remains as an interesting open problem to consider the generalization of some of the results obtained here (in particular, in relation with the scattering problem) to arbitrary N_c. Although the corresponding evolution Hamiltonian is known, and so is also its action in terms of Poisson brackets (cf. Sect 2), we expect the corresponding analysis to be complicated by the issue of the ordering of the operators. This is a new type of problem, which had not been encountered before in the framework of the color glass formalism, and so far it is not even clear whether this formalism can be extended to account for the non–commutativity of the color charges (see, e.g., the discussions in Refs. [8–11, 18]). One may however expect to be able to establish more explicit connections to previous approaches within perturbative QCD, which are aiming at the direct calculation of the $2 \to n$ gluon vertices at high energy and for arbitrary N_c [24–27].

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